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THE ROLE OF MATHEMATICAL AESTHETIC IN NETWORK-SUPPORTED GENERATIVE DESIGN: A CASE STUDY

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THE ROLE OF MATHEMATICAL AESTHETIC IN NETWORK-SUPPORTED GENERATIVE DESIGN: A CASE STUDY

by

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Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

May 2007

THE ROLE OF MATHEMATICAL AESTHETIC IN NETWORK-SUPPORTED GENERATIVE DESIGN: A CASE STUDY

Publication No. ___________

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The University of Texas at Austin, 2007

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Use of a next-generation, classroom-based network technology for mathematics instruction illuminates possible connections between the aesthetic perceptions of mathematics and mathematics teaching practices. Generative activity design makes use of participatory classroom simulations with the technology to allow students to fully engage in the activities from various levels and trajectories of understanding. Moreover, the student engagement with these activities produces artifacts, the projections of which make mathematical aesthetic visible and a substantial topic in the classroom discourse. This investigation entails the study of one secondary mathematics teacher, examining her instructional practices in the context of a networked-supported environment. This case study, conducted within the framework of a design experiment, uncovers the ways in which the teacher's mathematical aesthetic perceptions acted to (1) constrain her process of generative activity design and (2) frame her role in the mathematical discourse during classroom implementation of the network. Findings suggest the need for augmentation of a generative activity design framework to include overt connections to aesthetic.

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CHAPTER ONE

INTRODUCTION

> *When artistic objects are separated from both conditions of origin and operation in experience, a wall is built around them that renders almost opaque their general significance, with which esthetic theory deals.*
>
> John Dewey*, Art as Experience, 1934a, p. 3.*

Mathematics is commonly touted for its utilitarian advantages (Davis & Hersh, 1981); yet, there are some for whom the study of mathematics represents a more profound appeal, deeply rooted within artistic passions of human experience (cf. Brown, 1993; Tymoczko, 1993; Winchester, 1990a). Upon exploring the possibilities of the existence of the "mathematical unconscious," Papert (1980, p. 190) revives a stimulating topic first put forward by the nineteenth century French mathematician, Henri Poincaré. Papert asserts that the practice of mathematics is essentially a creative endeavor, which comports with Poincaré's provocative conjecture that "the distinguishing feature of the mathematical mind is not logical but aesthetic" (Papert, p.190). This assertion refers to an inherent characteristic of mathematics from which mathematicians, some mathematics educators, and a seemingly small minority of mathematics students derive some pleasure that emanates from a source of artistic-like expression in mathematics. For the mathematics education community, Papert and Poincaré's claims prompt oft-ignored questions about whether the notion of mathematical aesthetics can be framed in terms of teaching and learning context: What is the nature of mathematical beauty represented in educational settings? What educational affordances result from the influence of various perceptions of mathematical aesthetic on instruction?

This thesis responds to such questions by staking out an ideological stance on the contemporary educational practice first articulated by Dewey (1934a), who argued that artistry and aesthetic be retained and sought explicitly for the purpose of thorough understanding in any particular field of study. For Dewey it was the work of theorists "to restore continuity between the refined and intensified forms of experience that are works of art and the everyday events, doings, and sufferings that are universally recognized to constitute experience" (p. 3). Accordingly, it is the phenomenological experience of mathematical aesthetic, framed as an affective instructional goal in mathematics education, which is the central focus of this investigation.

Overview

This study employed case study methodology in the context of a larger design experiment to investigate the possible roles that mathematical aesthetic perceptions play in designing and implementing mathematics instruction with next-generation classroom network technology. In implementing network-supported generative activities in several secondary mathematics classes, it was discovered that a significant part of the discourse revolved around the mathematical aesthetic perceptions of the teacher. In many instances described herein, the teacher's sense of what she (or other participants) thought of as mathematically beautiful in the student solutions and artifacts, both individual and whole group, was as much a part of the conversation as her perception of mathematical validity.

Pursuant to this preliminary finding, this investigation had two goals: (1) to characterize the process by which aesthetic considerations emerged as part of the design constraints and guidelines for classroom implementation and (2) to examine the resulting influences of aesthetic—as it seemed to be reified by the network technology—on classroom teaching practices in the local educational setting. This study used a conjecture-driven research design (Confrey & Lachance, 2000), accompanied by "theory and common, core, classroom conditions in order to create and investigate new instructional strategies (p. 231)." The study was framed by the conjecture that HubNet technology (Wilensky & Stroup, 1999) initiates a type of mathematical thinking that emphasizes the artistic quality of mathematics practice and engages a strategy for mathematics instruction, informed by perceptions of mathematical beauty.

In this research project, the researcher collaborated with one secondary mathematics teacher in designing, planning, and implementing lesson plans that utilized a network-supported, generative teaching technology. Meeting over an eight-week period, the teacher and researcher discussed, refined, and tested her ideas for the most effective use of the classroom network in her Pre-algebra and Algebra II classes. The teacher acted as a partner in this research endeavor, taking a prominent role in the designing of lessons and also, participating in the analysis of their effectiveness—consistent with the designed experiment methodology. The lessons created for this network technology were modeled on a generative activity design framework proposed by Stroup et al. (2005; 2007).

This thesis examines some ways that aesthetic perceptions of mathematics help to provide structure for the social sphere of learning (mathematics structuring the social sphere (MS3)) (Stroup et al., 2002; 2005) within the process of designing and implementing a networked-supported learning environment. Empirically, this thesis considers the "work" done by the aesthetic perceptions of a secondary mathematics teacher in the context of her classroom teaching practices. Chapter Two examines network technology as a way to support generative activity design and understand the dialectic nature of the relationship between mathematics constructs and pedagogy, through which the notion of aesthetic may be empirically framed. The literature reviewed in the chapter impels a hypothesis that mathematical aesthetic can serve as an evaluative standard and design constraint for pedagogical practices in the context of a classroom-based network technology, because the technology makes the aesthetic visible, revealing it as a potential factor in network-supported generativity.

Chapter Three presents a detailed account of the case study process (conducted in the context of a design experiment) used to collect and analyze data, tested against the initial conjectures about network-supported generative activity design. Together, the teacher, the research setting, and the technological innovation define the case. This one context made sense as a candidate for this particular study, because the technology produces visible artifacts of the teacher's pedagogical decisions. These artifacts, projected images of the students' work (individual and collective) that accounted for a significant portion of the data set, were central to the instructional decision-making process. The researcher's attention was focused mostly on the teacher's (and students') ways of responding to these class-generated artifacts, which often took the form of emotional outbursts and physical gesturing like "high-fives" (hand clapping) or specific language and words like "cool" or "interesting". Such physical manifestations were taken as indications of an operationally defined notion of mathematical aesthetic.

Consequently, this investigation proposed to answer the following research question: To what extent does the teacher invoke mathematical aesthetic perceptions in designing and implementing networked-supported generative tasks?

Findings from the analysis of this single case, as presented in Chapter Four, revealed a scalable aesthetic quality in the teacher's mathematical thinking that resulted in particular modes of classroom activity engagement and discourse. This interpretation, moreover, compelled an extension of the generative activity design framework proposed by Stroup et al, (2005; 2007). While this study aims to help alter the tenor of discussions in mathematics education by revealing the artistic nature of school mathematics fostered by network-supported generativity, in Chapter Five it is argued that the findings may also serve as a basis for further conjectures about aesthetics that are broader in scope. It is further argued that the findings and interpretations of the case have implications for mathematics educational practices, in general, and not just in technological contexts. Research along this line is necessary in order to (1) more fully characterize aesthetic influences on teaching and learning of mathematics, (2) propose revisions in traditional task analysis frameworks with the aim of projecting aesthetic overtly, (3) leverage the connoisseurship and creativity of mathematics educators as evolving through practice, and (4) broaden the expectation that mathematics education should be only about that which is practical or applied in order to be compelling.

CHAPTER TWO

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

This chapter presents the conceptual framework for a design experiment to answer an important question about the nature of the mathematics in the process of implementing a novel technology for mathematics education: What aesthetic qualities exist in the mathematics classroom in a network-supported atmosphere? At the heart of this question is the interplay between content and pedagogy. The study was planned to examine this interplay through a dialectic analysis of mathematics and mathematics pedagogy as proposed by Stroup et al. (2002; 2005). The dialectic initiates a dialogue about mathematical structures and the social structures of mathematics pedagogy in a way that generates new conceptions of mathematics learning and new opportunities for students to engage creatively in mathematics classrooms. Mathematical aesthetic is a part of this dialectic and a theoretical lens through which to critically examine the mathematical activity and social structure in a network-supported classroom.

Before considering this dialectic, this chapter begins with the historical developments that led to the novel classroom-based network used in the study and explores the features of this innovation that reveal a unique interaction between content and pedagogy. Next, the various other educational contexts that instantiate a dialectical relationship between the mathematics content and mathematics pedagogy are examined. In the final section, the notion of mathematical aesthetics is considered as a possible strand in the dialectic and hypothesized as to the extent to which it may be seen to structure the social space of the classroom.

Network Technology

A new generation of classroom-based network technologies for science and mathematics learning has the potential to illuminate different perspectives of the content and pedagogy in teaching science and mathematics. The general architecture for these generative network designs includes individual devices (e.g. handheld calculators), a central computer to support real-time interactions between agents (e.g. peer-to-peer or whole-class), and a combination of public and private display spaces (e.g. calculator or personal laptop view screens or computer projections). This technology is next in the line of succession of technological endeavors to leverage and harness the full potential of classroom connectivity and social interactions.

Historical Overview of Classroom Networks

A review of the literature shows that this new network-based technology has archetypal roots in the historical development of mathematics and science classroom technology. The decision to use technology in a mathematics classroom setting is, first and foremost, an educational decision guided by the need to meet educational objectives. Nevertheless, many design technologies for educational settings, have been a "retrofitting" of technology previously or currently designed for purposes other than education (Kaput, 1992, p. 547). The best example of this is standard computer technology, which is found in many classrooms and utilized with only limited success, because educators lack a coherent pedagogy that integrates it into curricular objectives.

As more pedagogically sound computer software is developed, teachers are forced to shift their thinking about the nature of mathematics and mathematics teaching, in addition to having to develop their proficiency and expertise with new technologies. In little time technological designs and software have moved away from the pedagogically limited forms of skill and drill software, rooted in the behaviorist tradition, to those with less pre-scripted pedagogical forms like *Geometry Supposer* (Schwartz & Yerushalmy, 1985). *Geometry Supposer* was designed to help students test and verify their conjectures, compelling teachers to acknowledge student-generated claims and conjectures as viable mathematics objects in the classroom discourse. Kaput (1992) explains that using *Geometry Supposer*, "the locus of social authority becomes diffused; provision must be made for students to generate, refine, and prove conjectures; the teacher must routinely negotiate between student-generated mathematics and the teacher's curricular agenda" (p.548).

Shifting their approach to designing, innovators have begun to rethink the "retrofit" model for classroom technology, and instead are creating technologies to fit with pre-existing pedagogical frameworks. Responding to criticism that much of the traditional silicon-based technology used in classrooms is simply a tutorial design model ineffectively used for an entire class (Stroup et al., 2002), some have begun to view new networked-based technology as classroom specific (Abrahamson, 1999). Networks are taken to refer to silicon-based technology that allows students to communicate artifacts of their thinking with the rest of the class via handheld, laptop, or desktop computing devices. Use of classroom network technology represents an attempt to use the advantages of a social space such as a mathematics classroom to explore mathematics phenomena in more complex and dynamic detail. There is the potential with network technology to engage the group as more than the simple sum of individuals. In addition, it facilitates a connection between the pedagogical needs and possibilities associated with the class as a well-defined group and the design of the learning and teaching technologies.

One such technological innovation is the Classroom Communication Systems (CCS), calculator or computer networks that incorporate a Socratic method of questioning. Software that runs on the CCS, like *Classtalk* (1997), gives the teacher a chance to assess student understanding by interspersing the class lecture or activities with polling-type questions, thereby making the classroom more interactive. As histograms of the responses of the entire class are projected on an overhead screen, students can get an idea of how their answers compared to others in the class. Researchers found that teachers used the CCS system to assess students' background knowledge and understandings and to provoke class discussions (Abrahamson, 1999). While these activities were a step in a new direction for classrooms, the use of network technology was still in an infant stage in its development. Further, the use of the Socratic method as a pedagogical basis for networked activities may in fact be simply another attempt to scale-up a tutor model for teaching an entire classroom; that is, such a model does not fully address the nuances of the dynamic social spaces in classrooms. Questions are still evaluated in oversimplified terms of right/wrong.

While much of *Classtalk*'s accompanying pedagogy was dedicated to assessment of students' thinking, the full capacity of this classroom network technology was far from being realized. The relative limitations of network technology, in general, limited educators' thinking about more content specific uses. However, as newer generations of CCS technology become more readily available, researchers have begun to study in earnest the cognitive affordance of this design for the classroom and pushing its limitations as a pedagogical centerpiece and curricular workhorse (Kaput & Roschelle, 1996; Stroup et al., 2002; 2005). Newer utilities of classroom networks, employing the social structure of the school settings by supporting more complex inputs (e.g., "4*x*" or "2*x*+2*x*" versus multiple choice responses), attempt to leverage the diversity and large class sizes of typical mathematics classrooms. *MathWorlds* (Kaput & Roschelle, 1996), for example, is calculator software with features for sharing information from one student calculator/computer with the rest of the class. The new improvement in networking was focused on the exchanging of mathematics artifacts between students (as opposed to simple multiple choice inputs, "a", "b", "c", or "d"). With this application students interactively engage the technology to make sense of fundamental calculus ideas. *MathWorlds*, moreover, has similar effects as the original CCS designs in that students' thinking is made overt and is informed by increased interactions in the classroom. One primary difference, however, is that this technology is used to address crucial pieces, if not, the heart of the mathematics and science curricula in a more pedagogically sound way, extending the objectives of educational reform efforts mentioned in NCTM's (2000) *Principles and Standards for School Mathematics*. Another, more fundamental difference is the ability of the technological design to project an individual's thinking to the entire class. Thus, information is directed in many different directions, more accurately reflecting the nature of social settings that exist in the classroom.

HubNet

HubNet (Wilensky & Stroup, 1999) refers to a general network technological architecture in where a central hub distributes and collects information to and from the nodes of the network. The nodes may be single handheld devices or individual computers, one per student. HubNet supports the active engagement of all students in science or mathematics class through role-playing activities (as in participatory simulation that is mentioned later). HubNet may also become the central area in which to mediate further class activity and dialogue, as it serves as a common venue in which to project student-generated constructs. HubNet creates a physical space to serve as a proxy for an abstract mathematical space that allows for broadcasting diverse student thinking and ingenuity.

NetLogo (Wilensky, 1995), a parallel modeling and simulations environment, is a key component of the HubNet technology used in the present study. *NetLogo* acts as the hub for the network, collecting and displaying all of the students' activities and interactions as part of a live simulation. Wilensky (2001) showed that the use of multi-agent modeling languages like StarLogoT and *NetLogo* helped students develop a deeper understanding of more complex scientific systems like emergent phenomena. Combining *NetLogo* and a network platform (HubNet) developed by Wilensky and Stroup (1999), provided a framing metaphor for thinking about phenomena from multiple perspectives, namely agent (individual) and aggregate (whole-group). This environment supports thorough and deeper student understanding of how interactions between many objects at one level could lead to observable complex patterns at the global level. Students are able to send individual objects to the network that is projected from the teachers display as described in Figure 2.1. (Figure 2.1 is based on a reproduction of a figure found in Stroup, Ares & Hurford (2005, p. 184).)

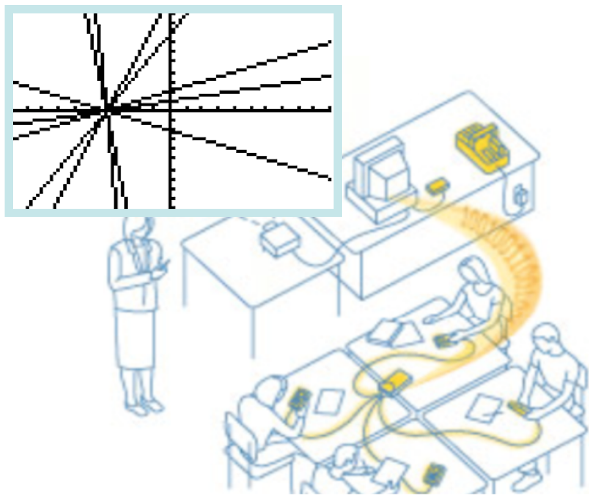


Figure 2.1 *NetLogo* Classroom schematic supporting student interaction aggregated in a central hub.

The design in Figure 2.1 is referred to as a HubNet and enables the classroom students an opportunity to control the direction of the discussion and to view an aggregation of the classroom gestures and artifacts on the central hub displayed at the front of the classroom. Teachers are able to use student-generated constructs visible to the whole class, ask questions about what they are seeing, and even challenge the responses. The aggregate of student work becomes a mediating artifact in the classroom. The discussion often is the object of the class discussions, both at the whole-class level and the small group level. This feature of the HubNet, in particular—the projection to the up-front space—causes certain qualities of mathematical and scientific ideas to be made visible and salient that may otherwise go unnoticed or rarely seen. This pedagogy has possible implications for the more affective goals of educational reform. For example, the beauty of mathematics and science ideas is one such quality that is especially highlighted by this technological innovation.

Participatory Simulations (Wilensky & Stroup, 1999), a pedagogical application of Hubnet, are live interactive simulations that use the HubNet architecture to capture individual and whole-class thinking about particular mathematical ideas that are projected for the entire class as depicted in Figure 2.1. Participatory simulation activities enable each member of the class to assume individual roles, the aggregation that encourages interesting patterns and behaviors of a complex system to emerge. The interplay between an individual's actions and that of the whole class becomes the source of the classroom discussion over the intended curricular topic. These activities, aimed at science and mathematics topics, are not designed to teach rote skills per se, but rather provide an experiential and conceptual basis for the key parts of the curriculum.

Multiple activities that use this hub concept, including *Regression*, *Disease*, and *Elevators*, have been developed (Stroup & Wilensky, 2004). Another of their creations is the *Function Activity*, a participatory simulation that involves moving a given point to a location on the x-y plane in response to a rule given by the teacher. In this simulation students are also able to send equations or analytic expressions to the network. One of the goals of the activity is to help the students get used to the convention of the Cartesian coordinate system and explore different conceptions of function and functional equivalence. This activity was the focus of the present study.

All of these activities were designed to elicit more student contributions. As students seek to make sense of mathematics concepts nested within these activities, they are empowered mathematically. Constructing direct connections between the mathematics and their own experience, students recognize their own intuition as potentially, mathematically valid. These network activities exhibit considerable potential to reorganize the social space of the classroom in such a way that it radically changes the roles of the teacher and students. Teachers must decide which student claims are mathematically valid (or invalid) and which are pedagogically worthwhile to explore. The net effect is that the teacher is forced to routinely negotiate between student-generated mathematics and the teacher's curricular agenda – after all, not all ideas are worth pursuing, even if time were unlimited. Such decisions are at the heart of a dynamical interaction of content and pedagogy that emerges in mathematics education.

Interaction Between Mathematics Content and Pedagogy

The tasks of implementing reform-based instruction emphasize the ways in which pedagogy gives rise to "real" mathematics problems. They do not require prepackaged knowledge, but a process-oriented mathematical knowledge that can be called upon in the course of teaching. Moreover, some researchers (Stroup et al., 2002; 2005) examine how teachers' mathematical thinking structures their classroom teaching as illustrateded in Figure 2.2. The conceptual framework of this body of research inverts the idea of pedagogical content knowledge (Shulman, 1987). As opposed to focusing only on the mathematics that emanates from specific pedagogical contexts, this research agenda also seeks to understand the various ways that mathematical perceptions themselves shape the pedagogical context.

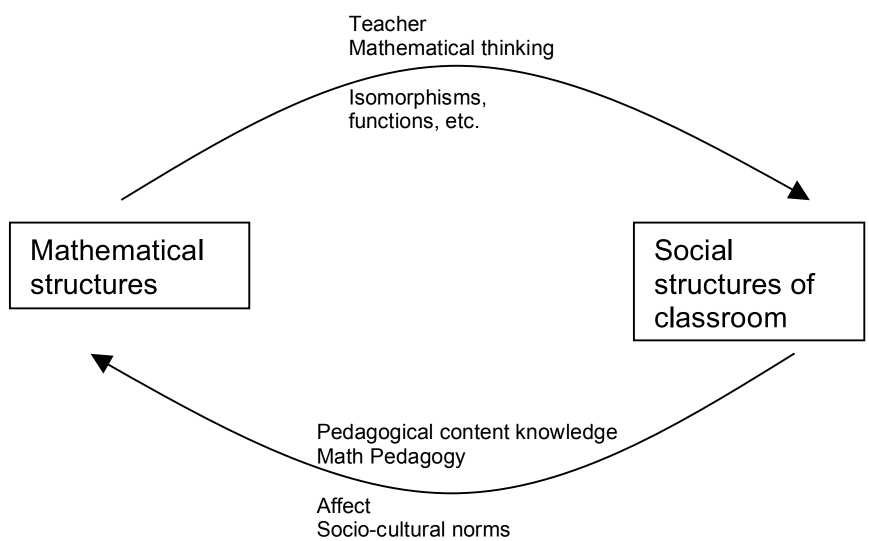


Figure 2.2 Content–pedagogy constructs arising from mutual influence between mathematics and classroom structures.

This body of work reveals the reciprocal influence and dialectic relationship between mathematical structures and the social structures of the classroom. Pedagogical practices, filtered through the socio-cultural norms of the classroom, clearly affect the mathematics that is learned, while the content knowledge of the teacher acts to shape the pedagogical practices used in the classroom, as depicted in Figure 2.2. The argument is that “dialectic is a kind of juxtaposition of ideas, often literally a debate, rather than a resolution of synthesis” (Stroup, Ares & Hurford, 2005, p. 188). Understanding “emerges via the activity of holding in creative tension even ideas that seem paradoxical” (p. 188). Mathematics structures and social structures of pedagogy are seen as *inputs* and *influences* on what happens in the educational setting. The dialectic is based partly on the assumption that "if mathematical and scientific structures are seen to fully participate in the social plane, then not only are they structured by the social plane but they also structure social activity, including learning and teaching" (Stoup et al., 2002, p.195). Recently referred to as Mathematics Structuring the Social Sphere (or MS3) (Stroup et al., 2002; 2005), the dialectic fosters a dialogue between the fields of mathematics and the socially-defined aspects of mathematics education, highlighting their influence on each other, whereby a careful reflection on the features of one gives insight into that of the other.

Critique forms the basis for much of the dialectical analysis between content and pedagogy, i.e., it is a social critique of mathematical structures and a domain-centered critique of social structures. Social critique of mathematics undermines the assumption that mathematics is static, monolithic and value-free. Examining mathematical practices in social contexts emphasizes the variety of choices offered to individual members of the community for ways to engage in mathematical activity. Stroup, Ares and, Hurford (2005) explain:

> For mathematics and science to be seen as socially significant and powerful, the question of what complex phenomena are worth investigating must be negotiated with students. For personal and collective agency to be advanced by a particular activity, design must more overtly attend to the following kinds of questions: Does this activity really matter, and for whom? How do the questions raised connect to students' history, culture, and community? In what ways do the activities facilitate students' development of meaningful ideas and insights in a way that can allow them to take action on their world? (p. 202)

The various mediating devices of the social sphere, namely language and other forms of communicating ideas exert substantial influence on content. Likewise, the variety of ways in which to participate in the social space of a mathematics class attest to the dynamic nature of content and imply that mathematics is, indeed, socially-constructed.

Conversely, a domain-centered critique of the social space focuses on the ways in which classroom activity is organized around domain-related ideas. Instead of seeing pedagogical decision about the social activity as separate from domain thinking, domain-centered critique of the social allows researchers to "think of mathematics as a socially structuring tool in learning research and design." (Stroup et al., 2002, p. 198) A good example of this thesis is displayed in the work of Kaput et al. (2002), described in Stroup et al. (2002).

> For example, in the curricular context of linear functions in y = mx + b (slope-intercept) form, to help understand the roles of the "m" and "b," each group of students is assigned its own value of "b" (the y-intercept), which controls the starting point of the group's "mascot" in a "race." In this race they are asked to finish in a tie with all the other groups' mascots at a given time (six seconds) and position (twelve meters). They then must determine the velocity (and hence slope) that accomplishes this task for their given initial conditions. (p. 199)

Here, plans for orchestrating social activity in the classroom are realized only in the context of mathematical thinking. This result stands in stark contrast to the notion that pedagogical decisions, like classroom management and grouping (e.g., ability grouping or small groups), are at a level separate and distinct from teacher thinking about (or with) content.

Examples of this dialectical relationship between content and pedagogy, as it relates, first, to mathematics teaching practices in general, and second, to the notion of *generativity*, provide support for the proposition of MS3 as a theoretical frame for instructional design in network-supported learning environments.

Content-pedagogy Interaction in Teaching Practice

A growing body of literature exists that describes the roles that mathematical knowledge plays in teaching, in which researchers closely consider and monitor the actual work of teaching to understand how mathematics is used in pedagogical contexts (Ball & Bass, 1999, 2000; Ma, 1999; Schifter, 1998). These studies call attention to the quality of teachers' mathematical knowledge in that they attempt to access not only "what mathematical knowledge teachers know, but how they know it and what they are able to mobilize in the course of teaching" (Ball & Bass, 1999, p.95). This research agenda assumes a domain of specialized mathematical knowledge for teaching that is qualitatively different from and extends beyond that required for personal understanding; however, the mischaracterization of this knowledge as simply a static set of facts does not describe the uniqueness of teachers' mathematical thinking. as that seems to belong to a distinct epistemological class, separate from that of other users and consumers of mathematics. "Mathematical insight, sensibilities, and knowledge" (Ball and Bass, 2000, p.89) are specific to the task of teaching.

The issue is not so much a question of "how much" as a question of "what kind" of mathematical knowledge is important for teaching. Reform-based mathematics pedagogy emphasizes the mathematics of classroom teaching practices. The construct of the quality of mathematical knowledge used in teaching has been elusive for researchers because "quality" resides squarely within the content-pedagogy gap. In their research Ball and Bass (2000) use constructs taken from the practice of pure mathematics to analyze episodes of classroom mathematics teaching and learning. Much of their mathematical analysis of classrooms is centered on the idea of resolving mathematical conflicts and reconciling "multiplicities," more than one representation or solution offered by various members of the class. Allowing for more than one viable solution is a major theme in reform-based teaching (NCTM, 2000; 1989). Ball and Bass (2000) write, "When more than one representation is offered for an idea or problem, examining the extent to which these are mathematically isomorphic, equivalent, or similar is crucial" (p. 199) and add, "The opportunity for this is actually ubiquitous in classrooms, where students' ideas contribute substantially to the enacted curriculum (p.199)." Yet, in order to perceive of student contributions in this way, teachers are likely to need mathematical sensibilities that are analogous to those of theoretical mathematicians. Such sensibilities, it can be argued, are a part of or point to the sorts of notions of mathematical aesthetic that are at the heart of this thesis.

Calling attention to the mathematical demands created by the context of mathematics instruction, Ball and Bass recount scenarios from classroom observation where such demands are manifested. For example, in a lesson in which the subtraction problem "32 -16" is given in the context of a story problem, "the students produced six mathematically distinct [correct] approaches" (p.96). This situation creates a mathematics problem itself, one to be solved in the course of instruction that demonstrates how the different student solutions are mathematically analogous. The different approaches offered by students in class reflect the various epistemologies and perspectives that students bring to class with them. In this context the teacher uses the practices of mathematics in order to understand the mathematics that is produced by their students. In this way teaching practice resembles that of mathematics. Ideas like proof, isomorphism, mapping between different representations, and equivalence classes, each a powerful mathematical construct in its own right, naturally arise within the settings of reform teaching, as discussed by Ball and Bass (2000).

Moreover, a core task in teaching is choosing between alternative representations of mathematical ideas that may or may not give a distorted view of the subject, emphasizing or de-emphasizing certain ideas. These perspectives seem to be of an evaluative or critical nature casting mathematics teachers as connoisseurs of mathematics and not simply as evaluations of procedural fidelity. The emphasis here is on "mathematical insight" or sensibilities that teachers need to create and evaluate mathematics learning environments. That insight is grounded in a nexus of ideas that is simultaneously pedagogical and mathematical.

Related to this notion is Ma's (1999) idea of Profound Understanding of Fundamental Mathematics (PUFM). She explains that PUFM is like a taxi driver's knowledge of a city; one can understand the connections from one location to another. PUFM is analogous to a map, a web of connected ideas, where links can be created from various directions and trajectories. Chinese teachers' attitudes and practices with mathematics are similar to that of mathematicians in ways called for by Ball and Bass (2000). They are concerned with mathematical justification for mathematics ideas that emerge within the context of teaching that involves discussing relationships between alternate solutions and representations. For these teachers, doing mathematics is a significant part of teaching. There is a sense in which the two, teaching and doing mathematics, are inseparable. Ma writes:

> It appears that a mathematics teacher should go back and forth between the two: doing mathematics, as well as making clear what it is that he or she is doing or teaching. Through this interaction, one develops a teacher's subject matter knowledge. (p.141)

Research along these lines does not necessarily purport to give a manual for reform teaching activities. On the contrary, this research does not even assume such a notion is desirable. Rather, the central issue is "the very nature of the mathematics of the 'reformed' classroom" (Ball & Bass, 2000, p.84).

With PUFM mathematics teachers understand mathematics constructs as a complex, connected web of related ideas. Unlike mathematicians who use these connections to create novel mathematics knowledge, teachers "reveal and represent them in terms of mathematics teaching and learning" (Ma, 1999, p.122). In this sense mathematics teaching is seen as an application of mathematical knowledge to pedagogical contexts. One facet of PUFM entails multiple perspectives of mathematical constructs. Multiple approaches or solutions to a single mathematics problem have differing trajectories in terms of their long-run pedagogical value. PUFM allows teachers to evaluate each of those strands for its cognitive payoffs. This feature of the teaching practice reveals a unique quality of the mathematical knowledge for teaching.

Likewise, Schifter, (1998) whose work is focused on describing the kind of mathematics knowledge required by and used in teaching, made an attempt to understand the manner by which that knowledge gets learned by prospective or in-service teachers. Her study, called Teaching to the Big Ideas (TBI), was done in context of a biweekly after school seminar for a project to enhance mathematics teaching. She detected phenomenological and epistemological shifts in teachers' conceptions of the nature of mathematics understanding, as well as what it means to do mathematics. As they "experienced mathematics, often for the fist time, as an activity of construction, rather than as a finished body of results" (p.65), Schifter found that teachers were beginning to see themselves as mathematical practitioners, specializing in pedagogical work. They also began to consider the mathematics put forward by their students as a mathematical field to be explored. The actual mathematics initiated in the context of teaching and learning is seen as a topic for "pure" mathematical inquiry open to all class participants, including teacher and students. Assessing student-generated ideas and evaluating their potential mathematical power, as reform-based teaching practices necessitates, are truly mathematical work. Teachers' mathematical thinking is organized according to the teaching context. In this sense, pedagogy is seen to structure the content in that it shapes what, when, and how mathematics is studied.

Examining the difference between U.S. and Chinese teachers, Ma (1999) found that Chinese teachers were better at conducting a mathematical investigation on their own and working more like mathematicians when exploring a novel student claim about basic mathematical ideas. She writes:

> The U.S. teachers did not show major deficiencies in their knowledge of topics related to the new idea. More than half of them know the formulas for calculation the perimeter and area of a rectangle. However, the U.S. teachers were particularly

> weak in their general attitude toward mathematics. Most behaved in an unmathematical way in approaching the new idea and did not investigate it independently. (p. 106)

The argument is that understanding the process by which mathematical knowledge is attained is a necessary pedagogical tool for understanding student-generated mathematics. This idea suggests that teaching mathematics well requires an acculturation to the discipline, and underscores the difference between simply possessing content knowledge as opposed to the specialized mathematical knowledge that teachers must have that mixes content with pedagogy. The argument follows that teachers' specialized knowledge and "general attitude toward mathematics" (p. 106) necessarily entail some form of aesthetic sensibilities.

This perspective acknowledges that teaching is the transformation, not the transmitting, of mathematical knowledge into forms that are accessible to students. Subject matter gets transformed through analogies, metaphors, activities and other pedagogical tools into instruction. Issues of pedagogy take on unique mathematical forms as teachers attempt to create reform-based learning environments from their perceptions and understanding of mathematics. For example, teachers' choices about the design of instruction and use of various representations or models of mathematics constructs are all rooted in their mathematical understanding. Wilson, Shulman and Richert (1987) espouse a framework for teaching that views pedagogical content knowledge (PCK) as a transformation for mathematics content. They write, "Influenced by both subject matter and pedagogical knowledge, pedagogical content knowledge emerges and grows as teachers transform their content knowledge for the purposes of teaching" (p.118). This idea suggests a mathematical function expressed as $\rho : K \mapsto K'$ where ρ is the pedagogical content knowledge function, K is the individual content knowledge of the teacher and K' is the pedagogical representation of that knowledge. In other words, K is mathematical content knowledge, the pre-image of a transformation under the ρ function invented by the teacher, and K', the pedagogical representation of that mathematics, is the resulting image revealed as instructional tasks. Enhancing this hybrid-type knowledge base for teachers seems key to enacting the affective aims of educational reform. PCK functions aside, values and attitudes about mathematics are also a part of what gets transformed in going from mathematics to pedagogy.

While the above examples clearly reflect a close interaction between content and pedagogy, highlighting the mutual influence that they have on each other, they center on ways in which one emerges from the activity of the other. The fact that more than one valid solution is presented in class gives rise to the notion of isomorphism. The fact that mathematics ideas have multiple representations affords alternative pedagogical approaches. What seems to be absent from these cases, however, is a sense in which the class activity is specifically and purposefully organized relative to implicit constraints determined by constructs of the domain. I argue that it is here, in these multiple approaches, that one might find some glimpses of the aesthetic perceptions of a teacher at work in pedagogy. As was pointed out earlier, not all mathematical solutions have the same explanatory power or elegance to which a teacher might want to acknowledge or call students' attention.

Generative Activity Design

Generativity, another instance of the MS3 dialectic, represents a domain-centered approach to designing instruction in a network-supported environment. The earliest notions of generativity, however, appeared as models of learning whereby knowledge growth occurs as the learner generates plausible connections between novel experiences and pre-existing cognitive constructs. "The essence of the generative learning model is that the mind, or the brain, is not a passive consumer of information. Instead, it actively constructs its own interpretations of information and draws inferences from them" (Wittrock, 1990, p. 348). Borrowing theoretical assumptions of constructivist learning theory, this idea casts learning as a naturalistic process that involves the active construction of interpretations and inferences by the learner. Generativity, then, is a cognitive process in which the learner's own initiative is used to create meaning from experiences, whether inside or outside the formal educational system. In a sense, generativity is a detailed description of the constructivist's account of learning. It provides an explanation of "the how" of constructivism. For each learning event, the learner generates a set of competing explanations, interpretations, summaries, analogies, or examples. In this way, generativity is an operational definition of constructivist learning, unveiling a cognitive mechanism by which knowledge is constructed by the knower.

The constructivist basis for generativity is most obvious in Piaget's (1980) version of constructivism, in which learning is described as an iterative process of perturbations and adaptations (via assimilation and/or accommodation) of existing cognitive structures. At first glance, Piaget seems to have used biology simply as a metaphor for learning.

However, in light of generativity, his model of cognitive growth and development has a remarkable resemblance to that of evolutionary biological processes.

Expanding on this biological theme, Schaverien and Cosgrove (1999; 2000) describe learning in terms of a generative heuristic, including the cyclical process of generating variant explanations for a new experience, testing them, and then re-generating those that survive the tests (g-t-r heuristic). They further relate the g-t-r heuristic of generative learning to neo-Darwinian synthesis, whereby survival is achieved through the selection of the "best" from a pool of generated variants. From this perspective, learning occurs through a process of selecting from variable or competing interpretations of experience. Notably, the role of generating and subsequently, selecting from alternative explanations or interpretations of experience in the learning process are central to learning.

At the group level, the g-t-r heuristic can be seen as a function of the generative activity of each student in the class, which is aggregated into one set on which the teacher can base pedagogical decisions. Instruction, itself, might be thought of as generative, and in this sense, generativity acts as a framework for activity design for group-oriented learning. The Cognition and Technology Group (1991) detailed several design principles for generative activities to support group learning. Generative instructional design, as a way to structure the social learning environment of the classroom, makes use of high levels of participation and production of the participants. As such, the Cognition and Technology Group lists examples of generative learning tasks that include writing, formal reasoning, explaining, formulating and/or solving problems. Their notion of generativity emphasizes learning that is contextual and authentic. The Cognition and

Technology Group further proposes two distinct design principles for generative learning environments, suggesting that teachers create environments that allow students, first, to generate mental representations (e.g., images) of the concept to be learned, and second, "[to] create shared environments that permit sustained exploration by students and teachers and enable them to understand the kinds of problems and opportunities that experts in various areas encounter and the knowledge that these experts use as tools" (Cognition and Technology Group, 1991, p. 35). Creating this shared environment that is open-ended enough to account for the generative activity of each individual in the class poses a major challenge for educators.

Stroup, Ares, and Hurford (2005; 2007) describe four features of generative design that meet the challenge posed by the Cognition and Technology Group, situating it within the context of domain-related constructs. They propose a design model to help guide the process of creating and engaging in network-supported generativity that is framed in terms of the dialectic relationship between content and pedagogy. The emphasis is on the prominent role of the networked to promote generative design as shown in Figure 2.3. (Figure 2.3 is a reproduction from Stroup, Ares, and Hurford (2005, p. 187).) Here, generativity is seen as extending the mathematics and/or scientific thinking of the teacher on to the social space of the classroom in such a way that content is used as a vehicle for organizing group-oriented instruction.

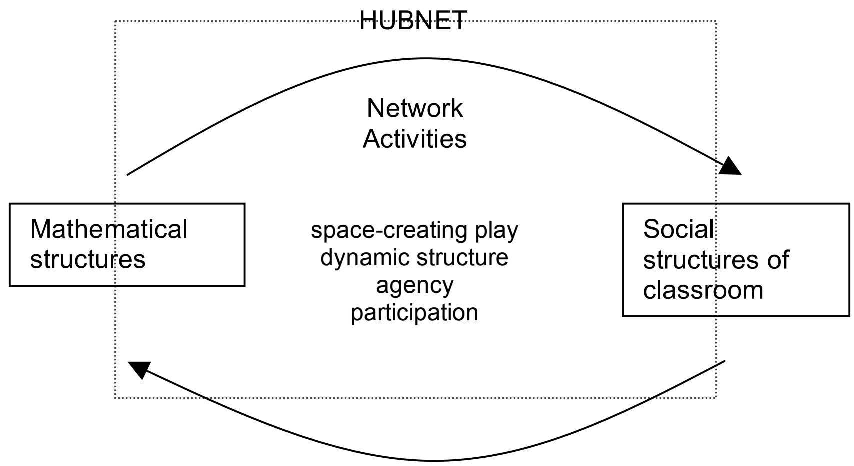


Figure 2.3 Features of generative activity design, supported by network technology and framed within the MS3 dialectic.

The first two feature of generative design, *space creating play* and *dynamical structure*, are exemplified by the following situation: all students in class are asked to create expressions that are equivalent to $2x$, instead of calculating, $x + x$. The simplified expression represents only one (perhaps even the trivial case) of a larger family of equivalent expressions. Generative teaching is shaping classroom activity in such a way that the students generate a collection of like mathematical objects that can be seen to fill a unified "space" or field. This feature of generative design is used to produce activities that are explained as, "overtly engaged with the sense in which learners are seen to be "playing" in mathematically and scientifically structured spaces. This play then creates a space of objects or emergent behaviors that embody students' understandings of the mathematical or scientific content, and they can serve as the objects of attention and analysis for the class" (Stroup et al., 2005, p.191). Instead of collapsing the pedagogical space to a single "correct" answer, as is traditionally done, this instructional design begins with a single mathematical construct and allows students to generate variants or examples for that construct.

By contrast, in a one-on-one tutoring model of pedagogy the question, "$x + x = ?$," is a sensible one, as this question has only one answer. However, in a classroom of 20 or more diverse learners, the question might be more effectively phrase as: "$2x = ?$." The number of correct answers to this question quickly diverges to infinity, depending on the creativity of the learners and the instructional skills of the teacher. Here, this domain knowledge transforms the pedagogical context, creating a broader space for more student participation. Specifically the teachers' domain knowledge is embodied in the generativity of the class. A single mathematical construct emerges from the generative activity of the students. In this, all mathematical objects that are 'the same as $2x$', taken in the aggregate, form an equivalence class. This idea has special significance relative to HubNet design, as it relies upon the projected image of the entire class' activity.

The other two generative design features proposed by Stroup et al. (2005) are *agency* and *participation*. If the whole group is projected in the up-front space in the classroom, then that projection becomes an important mediating artifact for student thinking and activity. They are afforded the opportunity to engage in the class on either of two levels, individual or aggregate. A student may associate himself or herself with a personally-created object on his or her own handheld device or in the up-front space as part of the larger collection in the projected image. On the hand, the entire class is challenged to collaborate in order to co-construct a single unified artifact or to fulfill a single class goal, clearly reflecting a group level of agency and various modes of *participation*.

Participation, as a feature of generative design, acts to situate student learning in a broader context, reaching outside the classroom. Stroup et al. (2005) write:

> A diversity of avenues of participation is available in network-mediated activity, including text, physical and electronic gestures, as well as verbal contributions to classroom dialogue (e.g., conjecture, prediction, observation, and explanation). Moreover, the collective character of participation in those modes of contribution, using a variety of representations of phenomena (texts, graphs, visual displays of emergent systems, language), and in inquiry-oriented discussion and analyses invites multiple ways of belonging. (p. 197)

The variety of entry points for participating in network-supported generativity suggests that students can engage in mathematics and scientific tasks with their own cultural identity and social practices that come from their home life. For example, calling out to peers on the other side of the classroom to direct whole-group activity or celebratory gestures after successfully completing group level tasks are not uncommon in a networked learning environment. These and other various forms of participating must be attended to and leveraged by the teacher.

What seems to be under-emphasized in Stroup et al.'s design framework is a notion of domain aesthetic, which may, in fact, emerge as a part of the considerations for generative design tasks relative to networks. The issue is especially salient for the HubNet because of its projection capabilities. Since a projected image exists as an artifact of the group generativity, this image embodies the mathematical or scientific construct, a domain aesthetic that has potential for emerging as a part of the conversation about how to organize generative activities in the classroom. How prominent is aesthetic in these activities? How consistent is the emergence of mathematical aesthetic (or the perception thereof) in designing for network-supported generativity? To what extent can mathematical aesthetic be thought of as an additional feature of the generative design framework proposed by Stroup et al. (2005)?

Mathematical Aesthetic and Generativity

This section briefly describes varying notions of mathematical aesthetic, relative to affective educational goals and other artistic experiences, based largely upon the writings of mathematicians, as well as mathematics education researchers Sinclair (2001) and Wang (2001). The section concludes by elaborating on conjectures about the work that aesthetic perceptions might do in teaching practices and anticipates what mathematical aesthetic might look like in terms of planning and implementing generative design activities for the HubNet technology.

Operational Characteristics of Mathematical Aesthetic

Educational researchers continue to have success in their quest to ground mathematics education research more deeply within the domain of mathematics itself (Ball & Bass, 2000; Ma, 1999; Shulman, 1987); however, a vital component remains absent from traditional research perspectives, namely the aesthetic of mathematics. School mathematics arguably includes some variant of aesthetic (quality notwithstanding), just as inherent in mathematics pedagogy, as in mathematics disciplines. Nevertheless, provocative ideas and titles such as *Humanistic Mathematics*

(White, 1993) and *Creativity, Thought and Mathematical Proof* (Winchester, 1990) bring the notion of aesthetic to the foreground without clearly defining mathematical beauty for the purposes of educational research or practice. The central claim in this thesis is that the perception of beauty in mathematics is a factor influencing mathematics education in ways that might be similar to the practice of mathematics. The issue is the work that a teacher's aesthetic perceptions do in instructional practices and the nature of the effect that mathematical aesthetic has on mathematics pedagogy. This supposition assumes that mathematical aesthetic perception, as a phenomenological experience, has observable manifestations and that they can be correlated directly to teacher behaviors and utterances.

Although much writing on the philosophy of mathematics suggest that "the wellsprings of mathematics are not utility and relevance, but in creativity, imagination and an appreciation of the beauty of the subject" (Whitcombe, 1988, p.13), there is no consensus on one single definition of mathematical aesthetic in the research literature. Its various definitions and explanations, however, leave little doubt as to its existence, at least as a phenomenological experience to a great many mathematicians. In describing this experience, Davis and Hersh (1981), for example, write:

> A sense of strong personal aesthetic delight derives from the phenomenon that can be termed order out of chaos. To some extent, the whole object of mathematics is to create order where previously chaos seemed to reign, to extract structure and invariance from the midst of disarray and turmoil. (p. 172)

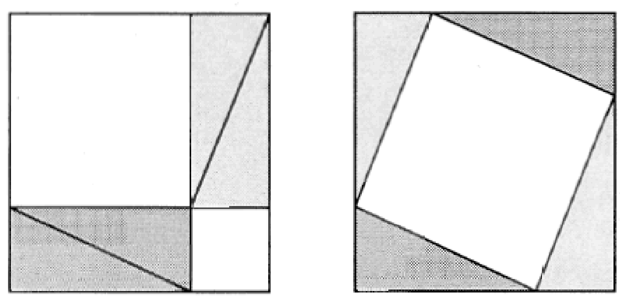
For Davis and Hersh, mathematical aesthetic itself is derived from the interplay between variation, randomness, and diversity on one the hand and singularity, precision and unity on the other. In this definition, the aesthetic of mathematics is found in the doing of mathematics. Tymoczko (1993), on the other hand, finds beauty in mathematical proofs, the artifacts of mathematics practice, implying that aesthetic evaluation of mathematics can be focused on the elegance and explanatory simplicity of mathematical arguments. This is consistent with Wittgenstein's (1956) language to describe mathematical surprise. He writes:

> One may see the value of a mathematical train of thought in its bringing to light something that surprises us—because it is of great interest, of great importance, to see how such and such a kind of representation of it makes a situation surprising; or astonishing, even paradoxical. (p. 143)

Here, the value of mathematics is connected intimately with its ability to "surprise". It is a critique of a mathematical explanation, much like an informed critique of artwork or music. In this case, as in the others mentioned here, mathematical aesthetic is cast as something to be perceived and engenders an affective response from the perceiver.

While much of the literature is preoccupied with constructing criteria for aesthetic judgments of mathematics, Sinclair's (2002) work suggests that aesthetic perception has functional characteristics relative to the practice of mathematics. Her explanation of mathematical aesthetic attends to the ways in which it might be integral to the process of mathematical knowledge creation. She outlines three different characteristics or "roles" (p. 12) of aesthetic that are all connected to some part of the process of mathematical inquiry: "evaluative" (p. 14), "motivational" (p. 39), and "generative" (p. 47). The *evaluative* characteristic is the most obvious of the three, because it assumes that the artifacts of mathematical work have varying quality and may be judged in a way similar to that of other art forms. In this characterization, mathematical aesthetic is seen as an inherent feature of mathematical constructs, like theorems, proofs, and solutions to problems, and the aesthetic of a mathematical artifact is of a certain quality. One such example is proof-by-picture of the Pythagorean theorem as in Figure 2.4 (Nelsen, 1993). It is specifically the elegance, simplicity and clarity of the explanations of this proof-by-picture, which contain so much of the aesthetic appeal for both mathematicians and mathematics educators.

The Pythagorean Theorem I



—adapted from the *Chou pei suan ching*
(author unknown, circa B.C. 200?)

Figure 2.4 Proof of the Pythagorean Theorem.

Aesthetic of proofs comes not only from the unexpected-ness of results (surprise), but from the elegance and economy of its explanation, and the inevitability or obviousness of the result. The objective of the mathematics doer is to create the best explanation with the least amount of required knowledge. The proof in Figure 2.4 requires little prerequisite knowledge of area of polygons to see that the *a*-square and *b*-square in the first picture has the same area as the *c*-square in the second.

The other two characteristics of aesthetic, motivational and generative, have less to do with the final artifacts of mathematical work and more with the actual process of mathematical discovery. The *motivational* characteristic refers to the initial attraction that mathematicians have to certain problems or facets of a problem. This attraction explains how aesthetic operates early in mathematical exploration, allowing for aesthetically-informed guesses about the possible directions for pursuing a mathematics problem. Further, the motivational characteristic of aesthetic also accounts for the manner in which problems are posed initially. That is to say, aesthetic perception motivates not only the direction an investigation takes, but also, more generally, which problems garner mathematicians' interests and attention. On this point, Sinclair (2002) writes, "aesthetic sensibility drives not only the unconscious choice that leads to mathematical discovery, but also the more general choices about which investigations to pursue" (p. 47). This point suggests that the motivational aesthetic also accounts for the variety of fields in mathematical study, algebra, analysis, and topology (including geometry), each having its own unique appeal to particular mathematical sensibilities. It is aesthetic that guides the choices about which mathematical avenues are interesting and worth pursuing.

In a related way, the *generative* characteristic of aesthetic, the most pertinent to this thesis on network-supported generativity, refers to the process of constructing mathematical objects within an on-going investigation. It is an attempt to characterize aesthetic as it operates in the process of mathematical invention. Like motivational, the generative characteristic of mathematical aesthetic assumes that a more elegant solution exists, and that mathematical exploration is pushed forward by the elegant combination of ideas. Sinclair's description of generative aesthetic is reminiscent of Poincaré's hypothesis of an "aesthetic sieve in the unconscious" (Sinclair, 2002, p. 103). Aesthetic is viewed as a mechanism for judging the trajectory of alternative solution paths, presuming that some are more fruitful than others, in terms of mathematical clarity and elegance. This characteristic subsumes evaluative aesthetic, suggesting that value judgments be made of possible solution alternatives based upon mathematical aesthetic, e.g., the more mathematically beautiful, the more mathematically productive. This idea resonates with the "multiple pathways, agreed upon endpoint" notion of generativity proposed by Stroup et al. (2007, p. 12). Here, each of the artifacts from a network-supported generative task is evaluated from the teacher's standpoint for its potential to promote deeper mathematical understanding. This point denotes the possibility of an aesthetic critique of each of the "multiple pathways" presented by students in terms of its pedagogical fruitfulness, and is further developed later on in the section, *Aesthetic in Generativity*.

While the various definitions and explanations of mathematical aesthetic attest to the experience that mathematicians have with aesthetic, the reifying of that experience into a useful educational tool is of concern in this thesis. The objective is not necessarily to present aesthetic as a well-defined construct, but to find ways it can be infused into educational settings. The challenge is in adequately explaining the phenomenological experience of mathematicians and reifying it into something to which others, non-mathematicians, can relate or, more importantly, find for themselves. These descriptions of mathematical aesthetic are presented, not necessarily to find a unified definition, but rather, to acknowledge its role in mathematics practice and ultimately to determine whether it does real "work" in the practice mathematics teaching.

Mathematical Aesthetic in Educational Settings

Several reasons exist for finding a connection between mathematical aesthetic and the process mathematics education. First of all, it has been argued that aesthetic is a natural part of the learning process (Dewey, 1934a). Dewey suggests that aesthetic is not a rarified experience, limited to a select few, but rather it is an integral part of the process of sense-making, accessible to all students. He writes, "Hence *an* experience of thinking has its own esthetic quality. It differs from those experiences that are acknowledged to be esthetic, but only in its materials" (p. 38). Explaining further, Dewey states:

> Nevertheless, the experience itself has a satisfying emotional quality because it possesses internal integration and fulfillment reached through ordered and organized movement. The artistic structure may be immediately felt. In so far, it is esthetic. What is even more important is that not only is this quality a significant motive in undertaking intellectual inquiry and in keeping it honest, but that no intellectual activity is an integral event (is *an* experience), unless it is rounded out with this quality. Without it, thinking is inconclusive. In short,

> esthetic cannot be sharply marked off from intellectual experience since the latter must bear an esthetic stamp to be itself complete. (p. 38)

From this perspective, aesthetic perception is an essential component of the process of inquiry, and thus, belongs to every domain, including mathematics education.

Secondly, the reform movement in mathematics education acknowledges that aesthetic is integral to the study of mathematics. This idea stems from a tradition of "looking to mathematicians and mathematics to provide insight on learners and school mathematics" (Sinclair, 2004, p. 1). The *Principles and Standard for School Mathematics* states:

> Mathematics is one of the greatest cultural and intellectual achievements of human kind, and citizens should develop an appreciation and understanding of that achievement, including its aesthetic and even recreational aspects. (NCTM, 2000, p. 4)

This is consistent with many mathematicians' own experiences with mathematical aesthetic (Davis & Hersh, 1981). Despite its reputation as a strict and rigorous discipline, mathematicians often speak of mathematical experiences as being filled of transcendent, powerful, astonishing, majestic ideas. This quote from Winchester (1990a) is a good illustration: "Mathematical thought in general is part of the dialectical thought of humankind and possesses the vagueness, the complexity, and the progress and regress of our thought in general" (p. v). The implication of his argument is that mathematics education, designed in a way that "demands student initiative, student independence, indeed creativity of both teacher and student in the mathematics classroom (Hersh, 1993, p. 15)," is likely to reflect an accurate representation of the domain.

A possible drawback of not making aesthetic a larger part of the conversation in school mathematics is that it can lead to a mischaracterization of the discipline, giving all but a small minority of students considerable misconceptions of what it means to do mathematical work. Conversely, making aesthetic part of the school mathematics curriculum has the possible effect of inviting more students to mathematics related fields. The promotion of mathematical aesthetic in school mathematics is part of an attempt "to discover the conditions under which mathematical activity can itself be intrinsically motivating" (Sinclair, 2002, p. 1). In this way, aesthetic is seen as an avenue for inviting traditionally underrepresented students to the conversations about mathematics.

I contend that this, in fact, relates to the *equity principle* of NCTM's (2000) *Principle and Standards*. It states that "[equity] demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students" (p. 12). I see aesthetic-based mathematics curriculum and instruction as both an accommodation for those students who are, for whatever reason, less inclined toward the study of mathematics and as an avenue to provide "access" to the appreciation of mathematical beauty. The argument follows that mathematical sensibilities and perceptions, as well as knowledge, can all be mobilized as tools for shaping students' experiences in mathematics.

Considering the role of aesthetic in school mathematics learning, however, prompts a series of questions that must be answered beforehand: In what ways are the phenomenological experiences of mathematical aesthetic reproduced in mathematics pedagogy? What instructional designs give students authentic mathematical experiences that include aesthetic appreciation? Pedagogical goals that are framed in terms of aesthetic presents unresolved issues, relative to connections between teachers' perceptions of mathematics and their teaching practices. There remains noticeable voids in our understanding about how a teacher's aesthetic perception of mathematics translates to an aesthetic experience for school mathematics students. The question here is specifically this: What instructional designs transform mathematics into pedagogy in such a way as to preserve it as an aesthetic experience?

Unfortunately, aesthetic has received very little attention mathematics education research. Lamenting the dearth of literature on aesthetics in mathematics education, some in the educational research community (Sinclair & Watson, 2001; Wang, 2001; Foshay, 1991) have begun to propose ways that aesthetic might be integrated as a part of school mathematics learning and instruction. Some of the proposals are especially relevant to network-supported generativity. Included in these is what Brady and Kumar (2000) describe as providing students access to "the motive of the inventor [of scientific and mathematics ideas]" as a way to access what is the essence of mathematical discovery and inquiry. They write:

> In using ideas to create new knowledge, the mathematician does not structure them in the same formal way that they would be structured for communication. In the process of the mathematician's learning something new, process and content are inextricably linked. What Clairaut called "the motive of the inventor" cannot simply be written down and learned from reading. Clairaut implied that mathematics cannot be adequately learned unless one is searching for and discovering mathematics in the process of working on problems. (p. 60)

This notion of motive is not far off from Sinclair's descriptions of motivational and generative characteristics of aesthetic. Emphasis on aesthetic as motivation in the process of mathematical work, gives student a clearer understanding of the discipline. It follows, then, that having a classroom dialogue that includes the aesthetic “motive of the inventor” is a way to model the practices of mathematicians. The more teachers are aware of the interconnections between mathematics and aesthetic the more likely they are to give students this opportunity appreciate mathematical beauty and to improve the pedagogical process as a whole by moving it closer to a truer representation of the domain and the work that mathematicians do.

Another suggestion for employing aesthetic in mathematics instruction is the use of visual artifacts to project mathematical truth and beauty simultaneously, not unlike Nelsen’s (2000; 1993) *Proofs without Words*. Aesthetic is projected through the visual representation of mathematics ideas. Sinclair and Watson (2001) also espouse the use of mathematical aesthetic through visual representations in education to inspire in students a sense of wonder about mathematics. Furthermore, they argue for the “structuring [of] students’ experiences of mathematics so that results which might otherwise seem commonplace emerge as surprising special cases (p.40).” Commenting on Fisher’s (1998) book *Rainbows, Wonder, and the Aesthetic of Rare Experiences*, they write:

> Attention to wonder provides a pathway to seeing something spiritual in learning about mathematical structure, in seeing mathematical results as surprises, in following up first instances of wonder with small steps of further wonder, in exploiting the visual and the intuitive, both fueling and being fuelled by curiosity. (p.40)

Sinclair and Watson assume "that 'wonder' being revealed through "the visual and the intuitive" are a propos to HubNet generativity, because of the centrality of the up-front projected image as a mediator for classroom activity and discussion of mathematics.

Aesthetic in generativity

Aesthetic is implicitly connected to some notions of generativity found in the literature. In particular, aesthetic is inherent in the notion of generativity proposed by Schaverien and Cosgrove (2000; 1999; 1997). For example, the selection phase of the g-t-r heuristic (describe earlier in this chapter) suggests that there are constraints and/or criteria for selecting from among competing interpretations. Furthermore, Schaverien and Cosgrove (2000) have already theorized that the entire g-t-r heuristic is set against a backdrop of a system of values, which serves to influence the kinds of variants that get generated and to constrain the selection of those that get tested. They write, "an evolutionary perspective [of learning] can account for the ingenuity of an individual (and even the apparently serendipitous coincidence of more than one individual) who generates a new scientific view, sometimes at a young age, which is of personal and subsequently of cultural value (p. 16)." It is in the context of culture and values that I find clear relevance of aesthetic perception. The task of selecting implies a sense of critique or evaluative processing.

The implications of aesthetic are more obvious in the notion generativity detailed in the generative activity design framework proposed by Stroup et al. (2005; b). They describe it as the "pathways and endpoints taxonomy of generativity" (Stroup et al, 2007, p. 9). They further explain, "Pathways are intellectual and/or routes for arriving at given endpoints. Endpoints are outcomes or artifacts created by learners that represent some form of completion of the generative task" (Stroup et al, 2007, p. 9). Because this framework of generativity elicits student-generated alternatives, it is, therefore, conjectured that aesthetic will emerge via student expressivity and teachers pedagogical use of it. Stroup et al's depiction of how generativity is expressed through "space-creating play" is indicative of a natural link between aesthetic and generativity. They write:

> *Expressivity* and *invented representational success*—including what we have come to call "space-creating play"—are highlighted in generative design over simply pointing-to-something-as-true across a wide range of forms of activity. The generative forms of space-creating play we present in our taxonomy are to re-frame, and serve to remind us of, the centrality that diverse expressive invention has in the on-going development of both content (what we learn) and pedagogy (how we learn). (p. 9).

Stroup et al. further explain, "with these sorts of 'multiple-pathway, agreed upon endpoint' tasks, better use is made of the uniqueness and creativity of each student in the class" (p. 12). It is predicted that aesthetic will emerge in the context of the teacher's role in defining and setting up the rules for activity engagement in the 'play' space. The following account of HubNet generativity provides a good illustration of the way in which aesthetic might arise as a part of the classroom conversation:

> For the '4x' activity 8x - 4x is certainly acceptable (and was praised by the teacher in Roxbury) as an example of and expression that is equivalent to 4x, but 1,000,004x – 1,000,000x and 100x/25 were seen as more interesting by the

> students themselves. We know this, in part, because once these particular examples were projected for the whole class, other students quickly worked on creating similar ones to share. Individualization is more naturally associated with seeing the uniqueness and diversity of each student's participation as making a vital contribution to developing sense-making of the group. Expressivity and inventiveness are celebrated and help drive the learning and teaching process forward. (Stroup et al, 2005, p. 13)

The above example illustrates that HubNet generativity provides an avenue for aesthetic critique to be a part of the discourse about the mathematics that emerges in the classroom. Teacher and students make qualitative evaluations of student-generated mathematics provides an avenue for aesthetic perceptions to interact with generative activity design. For some student-generated artifacts may be viewed as more "interesting" than others, and assuming that the teacher uses it in pedagogy in ways described by (Ball & Bass, 2000), it is in the "quality" of those alternative pathways that a sense of aesthetic may be invoked, either by the teacher or students (or both). As Stroup et al. (2007) explain, "the focus here is not so much on getting to an endpoint as exploring the 'quality' and kinds of possible pathways (p. 20)." The conjecture to be tested in this thesis is that the HubNet technology makes aesthetic overt and a significant factor influencing the teacher's decision-making process, as she chooses from among the various visual artifacts of student mathematical generativity through which to frame mathematical discourse in the classroom.

What has to be discovered is the criteria a teacher uses in evaluating the student-generated artifacts based upon their level of quality (high or low). Although Stroup et al.'s (2006; 2005) notion of generativity is slightly differently It is not a coincidence that Sinclair's (2002) taxonomy of the function of mathematical aesthetic in mathematics practice includes both motivational and generative features—the former influencing the initial choice of what mathematics problems to pursue, and the latter, the tedious process of mathematical inquiry...to find out whether it is connected in that aesthetic is seen to play a significant role in the generation of new mathematical knowledge and in selecting from among alternative paths of student generativity in the context of teaching practice, particularly if problems of teaching are expressed as mathematics problems (Carpenter, 1990). It is conjectured that close observations and analysis of teacher choices, behaviors, and utterances will reveal that the teacher's aesthetic perceptions of mathematics play a role in helping her realize the affective potential of student-generated mathematical artifacts. As this is not far from Sinclair's (2002) generative and motivational features of aesthetic in guiding mathematics practice, it begs the following questions: Does there exist analog of generative character of aesthetic in mathematical inquiry for the context of mathematics teaching? What is the role of the aesthetic perceptions of the teacher in selecting the mathematically valuable and interesting, both in terms of planning and real time decision-making in the classroom? What gets highlighted? Which student-generated constructs get talked about? While discussions of quality and creativity of student generativity may imply aesthetic, there is a need more empirical evidence to answer questions and characterize its role in instructional planning and implementation. How do pedagogical practices reflect the intuition and aesthetic assumptions in the mathematical thinking of the teacher? In what ways does aesthetic perception direct teacher exploration of different paths and trajectories generated by students in class?

Implications for research design

Because of the evaluative nature of this investigation of technological innovation, design experimentation seems to represent an appropriate methodology, in terms of its cyclical analytic process of making, testing, and revising conjectures. Hence, this thesis begins with an initial prediction about what aesthetic might look like in the context of classroom teaching practice. Yet, there remains one question of practical significance: What will serve as the object of analysis or constitute empirical evidence for aesthetic-like mathematical thinking? The research observations of some (Sinclair, 2002; Silver &Metzger, 1989) have uncovered a connection between aesthetic perception and affect through overt emotional responses to mathematical activity. Sinclair (2002) writes, "The essential emotional component of human aesthetic responses manifests itself physically: they (students) feel good; they provide exactly the sensations of pleasure described by some of the mathematicians" (p. 97). It is therefore proposed that emotional affect be a marker for the perception of mathematical beauty. In addition, the implications of Sinclair's (2002) work dictate that evidence of aesthetic perception be sought in terms of its generative and motivational features in the practice of teaching, which, together with affective cues, provides a way to bypass the ontological questions associated with mathematical aesthetic and instead, allow it to be reified as a "thing"—a tool for pedagogy. Evidence of the pedagogical function of aesthetic perceptions will, therefore, be sought in (1) teacher choices about which topics to discuss (vis-à-vis the technological innovation) and (2) teacher ways of talking about the mathematics that arises in the course of teaching practice.

This investigation is more than simply an evaluation of a novel technology, but a critique of the instruction and instructional setting that it fosters, from a perspective rooted in mathematics, proper. Levels of analysis of instructional designs, as well as the focus of that analysis take on a theme of educational critique. The questions being asked here are about the nature of school mathematics and its relation to teacher aesthetic perceptions. In what ways might one characterize the work that aesthetic perception does in implementation of HubNet generativity? In what ways might aesthetic present itself in the context of network-supported generative instruction, both in terms of planning and implementation? How does aesthetic perception contribute to (or detract from) the instructional process in the context of HubNet supported generativity?

CHAPTER THREE

METHODS AND PROCEDURES

The purpose of this study is to explicate the nature of mathematical aesthetic, as it is instantiated in network-supported generative design. Presented here is a case study that is situated as a part of a larger two-year design experiment (ISME, 2002), investigating the impact of the HubNet technology on classroom teaching and learning of mathematics. This case study comprises only a part of the data set of the experiment, and its purpose was to attend to the teacher, specifically and determine the extent to which mathematical aesthetics emerged as a feature of the teaching practices associated with generative network technology. Aesthetic perceptions of one secondary mathematics teacher were explored in the context of teaching practice, with special attention paid to the design and implementation of instruction that depended on the use of the HubNet. Specifically, the case study was to answer the following questions:

- To what extent does a teacher invoke mathematical aesthetic perceptions in designing HubNet -supported generative lessons?
- To what extent does a teacher invoke mathematical aesthetic perceptions in implementing HubNet -supported generative lessons?

As a result of the coding scheme used to analyze the data, the two questions converged into a third:

- How does the teacher mediate between various levels of activity engagement with the technology to invoke aesthetic perceptions?

This chapter outlines the research method used to answer these questions and to illuminate the role that mathematical aesthetic perceptions play in generative design. The first section is a description of the use of case study as a part of design experiment methodology. The next section is a detailed recounting of the data collection process and the nature of the collaborative effort between the researcher and teacher. The chapter concludes with a detailed description of the procedures used for coding and analyzing the data.

Case Study within Design Experimentation

Design experiments

Some educational researchers have called for the development of a methodology for studying educational innovation in a way that is analogous to that used in aeronautics, engineering, and other design sciences. Such a methodology, some (Collin, 1999; Collins et al, 2004; DBRC, 2003; Barab & Squire, 2004) have argued, could guide researchers in understanding how modifications of designed learning environments can affect certain dependent variables in the classroom setting. This work forms a growing body of research findings based upon design experiments (DE) or design-based research. A design experiment is a research methodology that documents the process of the implementing an educational innovation or intervention, detailing the successive revisions and improvements of the design, which were necessary to fit it successfully into particular contexts.

As design-based research in educational settings has matured into a viable methodology for educational research, the research community has begun to build consensus around a set if essential features DE's. The bulleted list below is a summary of those features:

- Design is tested in live classrooms.
- Many dependent variables associated with classroom context are analyzed.
- Characterize the independent variables that affected the dependent variables of interest.
- Revision of the design is flexible, not confined to pre-research plans.
- Focus on the complex social interaction of the classroom.
- Quantitative and qualitative data sources are drawn upon to develop a profile of the design in practice.
- Close collaboration between teacher and researchers in the design, revisions, and implementation.

A DE is more than formative evaluation of an educational intervention. The purpose of a DE is not simply to find out if an innovation is successfully implemented, but rather, to formulate or extend theoretical models of teaching and learning from the feedback in the implementation process. Barab and Squire (2004) argue this point, writing:

> What separates design-based research in the learning sciences from formative evaluation is (a) a constant impulse toward connecting design interventions with existing theory, (b) the fact that design-based research may generate new theories (not simply testing existing theories), and (c) that for some research questions the context in which the design-based research is being carried out is the *minimal*

> *ontology* for which the variables can be adequately investigated (implying that we cannot return to the laboratory to further test theoretical claims. (p. 5)

Design-based researchers are concerned with developing theoretical models of how and why a design works (or fails), not just describing its overall effectiveness. Moreover, design experiments often prompt refinements of theory, along with changes to the design itself.

It is the empirical evidence from the implementation process that informs theory, which in turn informs decisions about subsequent alterations to the design. It is this iterative cycle—design-test-theory-revision—that most characterizes design experiment methodology. This cyclical methodology sets up an interactive relationship between educational improvement on the one hand and theory development on the other. Testing the design in live classroom environments provides a consistent source of feedback in order to refine previous conjectures or theoretical models, which then leads to adjustments and retesting of the design. Theories that explain the success (or failure) of a design are formulated with respect only to the local classroom context in which it is implemented, and to that end, design-based researchers "work toward a theoretical model of learning and instruction rooted in a firm empirical base." (Brown, 1992, p. 143)

Integrated Simulation and Modeling Environment (ISME) project

The larger project from which this thesis emerges is the National Science Foundation funded ISME project (Wilensky & Stroup, 2002). The main project goal is to evaluate the use of the HubNet technology and participatory Simulations to help student understand complexity and emergent phenomena in science and mathematics.

Yet the project goals extend far beyond simple evaluation of network-mediated classrooms. Using design experiment methodologies, the studies that come out of ISME characterize the impact of the HubNet technology on the learning environment, often relying upon multiple sources of data and techniques of analysis from descriptive statistics to discourse analysis (Wilensky & Stroup, 2002). Operating almost exclusively in live classrooms around the country, much of the work is dedicated to building new theoretical models of learning, at the level of both individual and group. These models, as well as the HubNet activities themselves, are continuously tested and revised based upon feedback from classroom observations and interviews with teachers.

Case study

This thesis forms part of a larger set of data from a design experiment related to the ISME project. As a case study, it is meant to provide one level of analysis of classroom data. This is consistent with design experiment methodology, as case study is often part of the effort to get a comprehensive view of an innovation's impact on the functions of a 'typical' classroom. In her work Brown (1992) used just such a mixture of data collection and analysis. She explains:

> For example, in the initial study of reciprocal teaching, we provided pretest and posttest data on the 37 participation students, mini case studies on six children, together with transcripts from two children who differed considerably in how quickly they picked up the reciprocal teaching procedure, a precedent we have followed in more recent work. This approach enables us to see the magnitude of the effect in terms of outcome measures and to get a feel for the phenomenon

itself by looking at a particular child or group in depth. (p. 156)

While other parts of the ISME project were dedicated to studying learning, often using quantitative techniques of data collection and analysis (e.g. Hills & Stroup, 2004; Hurford, 2004), this thesis is a case study designed to attend to the teacher specifically, as she was necessarily a co-designer and played the primary role in implementing the technology in the classroom. In order to extend theoretical models about generative activity design supported by classroom connectivity, I contend that consideration of the unique role of the teacher is paramount.

Collection and interpretation of the data was informed by methods of what Stake (1995) calls "instrumental case study (p. 3)" in an attempt to develop a thorough description of the ways in which mathematical aesthetic perceptions of the teacher show up in the context of generative design and implementation of the HubNet. Yet, this study is not undertaken to understand this particular case, but to illuminate a broader concept that is exemplified by this case.

By paying close attention to teacher's utterances and pedagogical moves associated with designing for and implementing the technology and her reactions to the artifacts created in the network-environment, the issues surrounding this case emerged as a salient and seemingly important features of the process. The focus of the case analysis is mathematical aesthetic, as it is subjectively perceived by the teacher. From a phenomenological standpoint, I want to understand how the teacher's aesthetic perceptions are connected to the capacities of the HubNet technology and features of generativity. The aim of the case study is to provide the reader with a sense of the experience of perceiving aesthetic in the various pedagogical artifacts of the generative teaching technology, describing its varying qualities. Accordingly, I used aggregation of data collected from several episodes of the teacher planning and implementation to produce emergent categories of aesthetic. Analysis of this sort almost certainly requires the support of a qualitative methodology like case study, because the phenomena that this study purports to investigate, mathematical aesthetic perceptions, are inherently qualitative in nature and yet to be fully developed and explored. This assertion is supported by Creswell's (1998) description of the rationale for using qualitative research methodologies. He writes:

> Choose a qualitative study because the topic needs to be explored. By this, I mean that variables cannot be easily identified, theories are not available to explain behavior of participants or their population of study, and theories need to be developed. (p. 17)

Perceiving mathematical aesthetic, as a phenomenological experience, presents a difficult challenge for the purposes of empiricism. Yet, one operating assumption of this thesis is that observations of the teacher's behaviors and utterances, with respect to network-supported generativity can reveal the nature of her mathematical thinking, including aesthetic.

As well as being an instrumental case study, this research examines the impact of a design educational innovation on a live mathematics classroom, which adds sufficient complexity to warrant a qualitative investigation. In discussing the various functions of cases, Merriam (1988) writes, "The case study is a particularly good means of educational evaluation because of its ability to explain the causal links in real-life interventions that are too complex for the survey of experimental strategies (pp. 28-29)."

In the context of a live classroom, there are too many variables and factors, thus, making naturalistic inquiry, e.g., case study, a more appropriate methodology.

Although the case study did not contain the iterative cycles normally found in design experiments, the procedures for data analysis and interpretation used reflective cycles, starting with a theoretical notion of aesthetic and leading to refinements of that notion, as dictated by subsequent empirical analysis. As presented, this case is, among other things, an illustration of the close collaborative nature of teacher and researcher in design research and some of the issues that arise within this collaborative effort.

Research setting

In this study I collaborated with Martha, a secondary mathematics teacher, whose passion and aesthetic perceptions of mathematics first became evident to me during a summer workshop on participatory simulations. Over the subsequent two years Martha participated in the ISME project.

Martha taught at an inner-city charter school in the Mid West region of the country. The school had students from sixth grade to eleventh grade. The student body was very racially and ethnically mixed with students from varying socioeconomic backgrounds. Some students road the city buses from as far away as 15 miles, sometimes transferring one or two times, to come to the school. Other students, many of them Hispanic, came from economically depressed neighborhoods on the west side of the city. A few were there to take advantage of an alternative educational model that focused on art education, "first amendment" rights for students (ASCD, 2003), open student forums, and other non-traditional school activities. However, many students came because they were suspended from other schools for violence, drugs or other behavioral problems. This resulted in a surprisingly diverse cultural mixture of several hundred students, when contrasted with the district-wide demographics.

Martha's classes were relatively small, 12 to 16 students. They sat in groups of three or four students gathered around rectangular folding tables placed in various orientations around the room. Martha frequently made use of cooperative groups in her lessons, often setting the students to work on problem solving tasks for the majority of the class time. The Interactive Mathematics Program (IMP, 1999), the adopted curriculum of the school, is conducive to the cooperative group teaching strategy. IMP is a reform-based curriculum that emphasizes problem solving in mathematics by using narratives of mathematical situations and often requiring students to work cooperatively by sharing information.

The classes usually were a mixture of students in terms of age and ability level. Martha's IMP 1 course, with content similar to that for Prealgebra, was composed of sixth, seventh, and eighth graders. According to Martha, the younger students were often the brightest and more studious than the older ones. Her IMP 3 class, however, was somewhat more homogeneous with mostly eleventh graders and a few tenth graders. Because the class covered mostly Algebra II concepts, fewer students took it, as it was not required for graduation from state-funded schools.

Martha expressed some reservation about being the subject of a research study in mathematics education, as such a study could call into question the depth of her understanding in mathematics. Initially, she seemed to lack confidence in her mathematical knowledge. Although her background and schooling was in biology, the necessities of the schools in which she taught had, for much of her teaching career, demanded that she teach mathematics. Her current inner city charter school was no exception. In fact, since the charter for the school had been in existence, Martha had been either the only mathematics instructor or head of a three-person mathematics department.

Much of the powerful innovation in the school came from the co-founder/principle/superintendent, Tonia, also a mathematics educator. She and the rest of the faculty seemed, by in large, to reject the "traditional values" stigma of the community in which it was located, opting instead to be a First Amendment school **(**ASCD, 2003**)**. Tonia continually encouraged her mathematics staff of two and half accept reform-based teaching practices. Both Tonia's and Martha's attitudes about innovation in education made for a natural fit in this study on technology and mathematical aesthetic. Tonia and Martha were instrumental in implementing the IMP curriculum. Martha was also interested in using the HubNet technology as a way to visually represent the mathematics ideas presented in the curriculum and give students a different cumulative perspective of mathematics.

Data Collection

Collaboration

Over the course of eight weeks, the researcher and Martha interacted on approximately 10 different occasions either to discuss ideas for or to plan implementation of the HubNet participatory simulations in her seventh and eighth grade IMP 1 and eleventh grade IMP 3 classes. Most of the lessons were revisions or extensions of participatory simulations activities that appear in Stoup and Wilensky (2004). Others were HubNet versions of activities from Stroup (1997). Seven of the planning sessions were captured on videotape. Of the ones that were videotaped, five of the sessions were spent brainstorming ideas for or conceptualizing lessons that made use of the HubNet technology. The researcher and teachers collaborated to make changes and refinements in the lessons and create worksheets to help drive the lesson. During the other two videotaped sessions, they practiced the lessons with each using three or four calculators to act out the roles of the students, while the other used one other calculator to play the part of the teacher. Each of these planning sessions lasted anywhere from one to three hours, taking up most of Martha's planning period and sometimes extending to after school hours.

In addition to the lesson planning sessions, the researcher also participated in and observed the actual class sessions, five in all, during which Martha taught the HubNet participatory simulations lessons. The activities were run on a prototype classroom calculator network that required several computers; it was practically impossible for one person to instruct the class and operate the network. Martha conducted the class, while the researcher controlled the technology from the back of the room. The researcher took cues from Martha as to when she wanted the predetermined network programs launched, the projection of the whole-class screen turned on or off, or the entire activity reset. These data, both the classroom observations and the lesson design sessions, reflect the close researcher-teacher collaboration that is customary with design experiment methodology (Brown, 1992; Collins, 1992).

Emergent generative artifacts

In the course of the study we planned four different lessons based upon activities found in Stroup and Wilensky (2004) and Stroup (1997). As is the case with all generative activities, the lessons allowed students to produce some visible artifact of their thinking. These artifacts, which are referred to as emergent generative artifacts, are just as important a source of data as the interview/collaboration sessions or classroom observations.

In the *Rule for Points* lesson, Martha asked the students to move a point according to a rule. The students began with a single point randomly located in the x-y plane on their individual calculators. At the same time, on the overhead screen each student's point was projected as a specific icon. As the students moved their own points with the arrow keys on the calculator, they also moved the corresponding icon (avatar) on the screen visible to the entire class. With all the students' points shown together on the screen, Martha fixed a particular x-value for each student and then asked them to move the point to a new location where the y-value was one more than the x. (This rule was particular to this one class session. Other classes were given different rules.) Figure 3.1 provides a sample of the image that appeared for this rule.

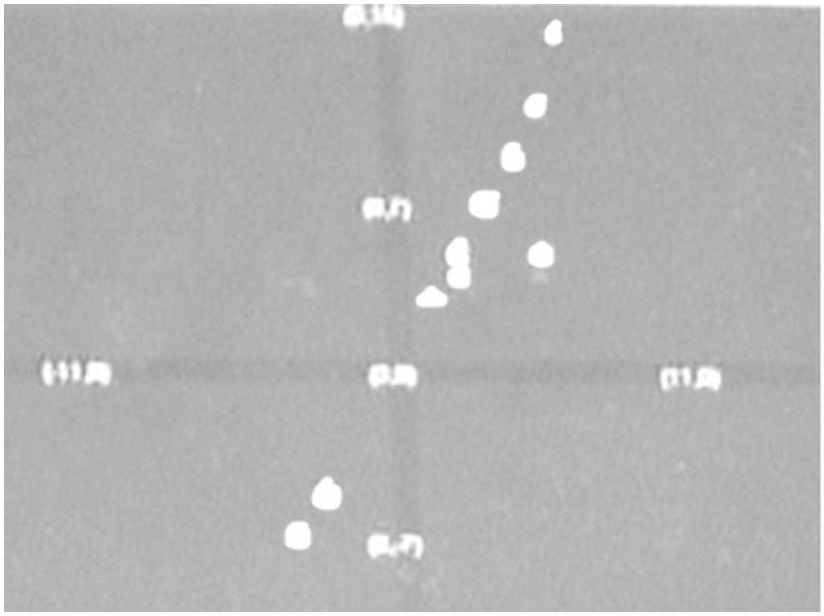


Figure 3.1 Still image from a video clip of the *NetLogo* screen appearing on the up-front display during the activity, “move to a place where your *y*-value is one more than your *x*-value.”

The *Linear Family* lesson began with Michelle displaying a single point on the *x*-axis for the class to view asking the students to create and send linear equations that passed through that point. Figure 3.2 is a reproduction of the whole-group artifact from this lesson. In this case the students were asked to make a linear equation that passed through the point (-4,0).

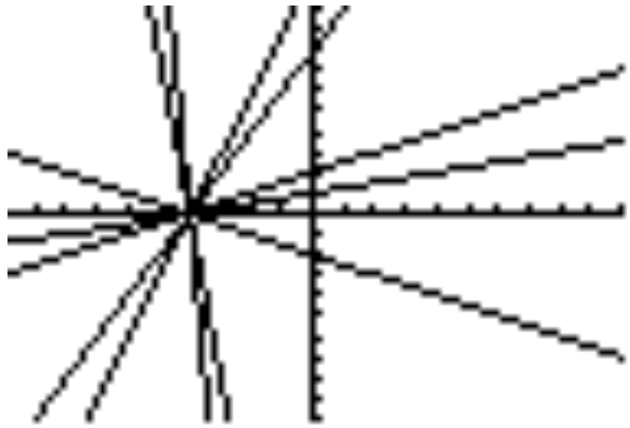


Figure 3.2 Calculator view screen, showing the likeness of image appearing up front for the challenge, “make a linear equation that passes through the point, (-4,0).”

The *Quadratic Family* lesson was an extension of the *Linear Family* lesson. Here, Martha asked the students to form products of two linear factors, one from each of two linear families and then send their equation to the network. Figure 3.3 is a reproduction of the whole-group artifact projected for all the class.

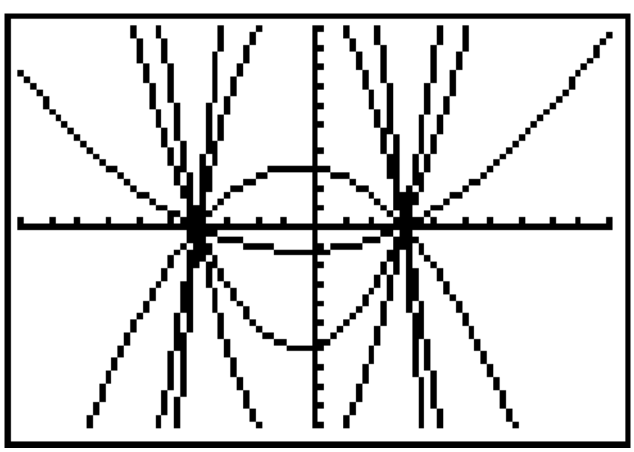


Figure 3.3 Calculator view screen, showing the likeness of image appearing up front after students graphed the product of two of their linear equations.

The *Product of Lines* lesson was an extension of the *Rule for Points* lesson. Martha first asked the students to move their respective points to a location that met the criterion she established by a linear rule—in this case, where the *y*-value was one more than the *x*. She then repeated the activity for a separate point and linear rule while the other points remained on the overhead screen. However, this time the students kept the their previous *x*-values and were only allowed to move the second point vertically in order to satisfy the second rule where the *y*-value was two less than the *x*. The result of these two activities was the formation of two lines (linear functions), comprised of all of the students' points. See Figure 3.4.

Afterwards, the students moved a third point (again with the same *x* value) to a *y* value that was the product of the previous two, forming a parabolic curve. Figure 3.5 is a reproduction of the whole-group artifact of the activity.

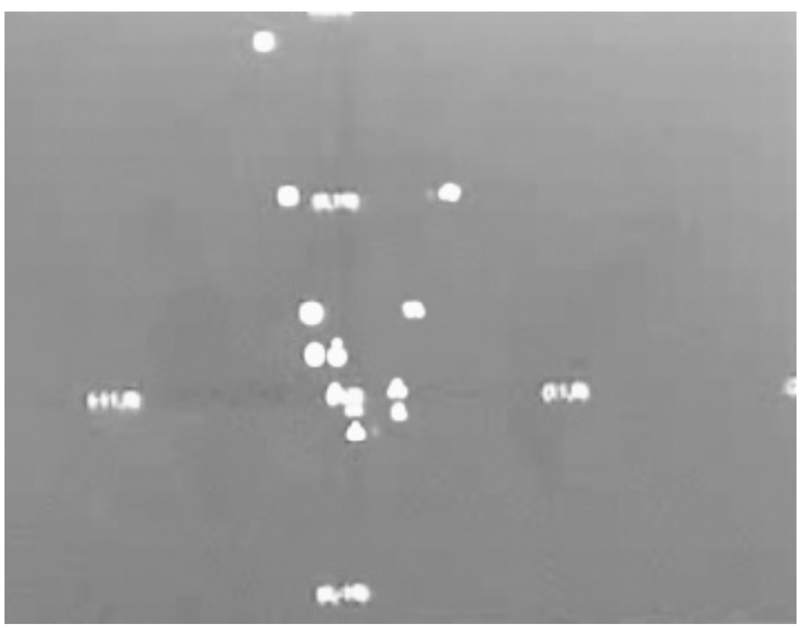


Figure 3.5 Still image from a video clip of the *NetLogo* screen appearing on the up-front display during the activity, "move to a place where your *y*-value is the product of your previous two *y*-values."

Coding and Analysis Procedures

To analyze the data, videotape data were first organized into stages of lesson development and implementation in what are term the *Conception*, *Planning*, and *Implementation* phases. In the *Conception* phase, the researcher and teacher discussed ideas for generative lessons, creating a sketch of the lesson design, making decisions about sequencing and layout of any worksheets, if necessary. The *Planning* phase was marked by instances in which both the teacher and researcher acted out the lessons with the HubNet, with each taking three or four calculators each. Finally, the live class sessions themselves comprised the *Implementation* phase. The application, *iMovie* (Apple Computer, Inc., 2002) was used to digitize all of the videotapes of the phases.

A video transcriber tool (Stroup, 2002) was used to review all of the video data and place beginning and ending time stamps on segments of the digitized video that referenced (according to the researcher's opinion) the aesthetic perceptions of the teacher or students. The transcriber tool, created specifically for the ISME project, is a script application that is programmed to link *Quicktime* (Apple Computer, Inc., 2002), a digital video software, and *Filemaker Pro* (Filemaker, Inc., 2002) database software. See Figure 3.5. While the video runs in *Quicktime*, the reviewer can activate the transcriber tool to stop the video, mark a time stamp, and create a record in a database in *Filemaker Pro* that attaches a top-level code to that section of the video, thus, creating individual records corresponding to segments of the interview/collaboration sessions and classroom observations.

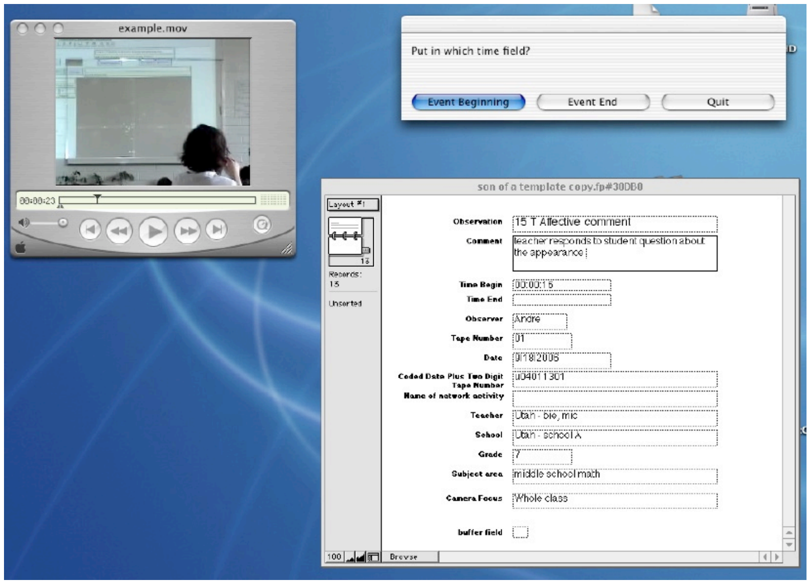


Figure 3.5 Video transcriber tool used to create database records from time-stamped video clips.

Each of the data records is a quote or chunk of dialogue (sometimes including non-verbal gestures) that centered on Martha's ways of commenting on or responding to others' comments about the artifacts produced in planning or in the classroom implementation of the HubNet lessons. One goal of this procedure was to describe the ways in which aesthetic continually showed up as part of her (and sometimes the students') ways of talking about the artifacts during the class, as well as in her planning HubNet lessons around their appearance on the up-front screen or personal viewing space on the calculator. In all, 208 aesthetic-related records (time-stamped video clips of one minute or more) were found that could be tied directly to the aesthetic perceptions of the teacher, as determined by the researcher.

The database was entered in tabular form for further analysis. First, each record in the data table was reviewed, along with the accompanying video clip. A description of each record was made in terms of one concept, a short phrase that summarized the exact nature of that particular quote or action. More detailed comments were made of many of the records, especially those that seemed to have obvious connections to the teacher's perceived aesthetic of a mathematical construct. Figure 3.6 displays a partial listing of the database table with the coding of the teacher "aesthetic-related" record as more completely shown in Appendix A.

Tape u03111801 (interview)

Time	Concept	Quote	Level	Description
4:25	Ideas for part sims	Using geometric patterns		
5:20	Focus on pattern recognition in activities (induction)	It's a pattern that now...	Agg?	Using patterns with many examples as a way to begin to think about part-sims and generative acts
11:11	Aesthetic appeal of pattern (rules)	Let's divide by two...	Agg	
11:28	"interesting" (marker for mathematical aesthetic)	Oh that's interesting...	Ag?	
13:40	Pedagogical defense (modeling math practice)	This first patterns unit is kind of to get them...to start looking at math differently		
22:07	Allusion to math utility	Maybe it's useful because...		
22:30	Aesthetic purpose over utility	A: I noticed that you chose to look for the patterns in the i's...	Ag?	Teacher makes decisions based upon aesthetic and utility of math
23:00	Ideas for part sims	A: so how can we turn this into a part sims?		Proposed interview question: What advantages of hubnet relative to that which you like most about math?
26:23	Aesthetic appeal of students' pattern recognition	We don't need the table...I'm doing this every time...	Agg?	
31:50	Ideas for partsims:	class creates a function of number of diagonals per polygon	Agg	
35:00-45:00	Joint problem solving between researcher and teachers (proof)	N lines creates how many partitions of space?		

Figure 3.6 Database table with coding of each "aesthetic-related" record.

Examples of the kinds of records that were associated with aesthetic were emotional outbursts, specific language and words like “cool”, physical gestures like “high-fives”, times the teacher called attention to specific student-generated artifacts that she thought were “interesting”, or times when she called attention to particular features of the projected whole-group artifact. Such examples were identified based partly on the work of Silver and Metzger (1989), mentioned in the previous chapter, who argue that emotion is a cue to aesthetic enjoyment in mathematics. Particular attention was also paid to the teacher’s reactions to students’ comments about the emergent generative artifacts produced by the HubNet lessons because they represented visual depiction of mathematical ideas. This focus was pursuant to the suggestions of some educational researchers (Sinclair, 2002, 2001; Sinclair &Watson, 2004; Wang, 2004) that mathematical aesthetic can be revealed through visual representations of mathematics. Teacher quotes and behaviors indicated her personal insights about mathematics constructs like proof and reasoning, influenced by mathematical aesthetic perceptions (Mack, 2002), were also captured. The teacher also made references to the aesthetic of student-generated artifacts from past classes. These pedagogical critiques were seen to reflect the teacher’s preferences for certain types of instructional designs and indicated her affective goals relative to mathematics curriculum and instruction.

Some records that were thought to pertain to the teacher’s perceptions of mathematics and mathematics pedagogy, in general, were also gathered, even though they could not be tied to aesthetic perceptions directly. They were integrated into the data set, however, because they helped to cast a more complete description of the case. These records included comments related to the creating of the lessons, the mathematics within the lessons, and predictions of how students would respond to these technology-driven lessons. Other records that were originally thought to be pertinent to the case were later discarded because they seemed to diverge from the central focus of the investigation. For example, an entire episode during the conception of a HubNet lesson seemed to say more about the teacher's notions of mathematics, separate from aesthetic perceptions, so it was not included in the findings overtly, but implicitly as a reference to the teacher's perceptions of mathematics in general. Also excluded from final data analyses were episodes in which the students exhibited obvious excitement, which was later found to be more about using a novel classroom technology than experiencing mathematical aesthetic. These observations were not recounted in detail because subsequent dialogue revealed that the affective responses of the students were not necessarily mathematics-related. Instead, these and other notable episodes were simply referred to in order to add a realistic texture to the accounts, as is necessary in describing a case (Stake, 1995; Creswell, 1998). This accumulation included many records (55), determined to be related to mathematical aesthetic perception, but not further analyzed or coded, because they did not pertain directly to the HubNet lessons or the generated artifacts. For example, in one interview with the teacher, while trying to decide on the topic for a HubNet lesson, the teacher and researcher talked at length about how her students "enjoyed" finding patterns in a data set as a way to prove a mathematics concept inductively, as opposed to simply being given a rule or formula. Aesthetic perception was both inherent and explicit in this episode, but not directly related to HubNet generativity.

On the second review of the data tables the records were marked as referring to perceptions of the pedagogical artifacts from either the individual (agent) level perspective or group (aggregate) level perspective. The former was coded “Ag” and the later, “Agg”. (Refer to Figure 3.6.) This coding scheme was informed primarily by Stroup and Wilensky’s (2003) notion of the “embedded complimentarity” of agent and aggregate-based reasoning in studying emergent phenomena, as represented by HubNet participatory simulations. Because the artifacts produced in the lessons were either whole group or individually constructed, they were perceived and referenced by the teacher in ways that were distinct and qualitatively different. This preliminary analysis is supported by the work of Penner (2000), as well as that of other researchers (Chen & Stroup, 1993; Resnick &Wilensky, 1998; Resnick, 1996; Lev & Wilensky, 2004), who describe the necessity of understanding complex phenomena from multiple perspectives. Moreover, Stroup et al’s (2005; 2006) discussion of the individual creativity and inventiveness, fostered by network-supported generativity, was also informative in establishing theoretical sensitivity to the occurrence of aesthetic perceptions of individually-created artifacts.

In all, 68 records referenced student-generated artifacts from the individual level of perspective and 42 for group level. Some (58) records, however, could be defined neither as entirely individual nor aggregate, but a combination of both. One of two special codes, either “Ag?” or “Agg?” was assigned to many (43) of these records. The former code was used to identify records in which the teacher referenced the artifact mostly from an individual level, but also from whole group within the same episode. The latter code was used for records in which she did the opposite, talked about the artifact mostly in terms of the larger picture generated at the group level, but also referenced individual objects within a single quote or action. A small group of the ambiguous records were assigned to multiple classes.

During the third examination of the data tables, the records were reorganized and grouped according to the lesson and phase in which they occurred, in order to find patterns associated with phases of lesson development. Additionally, this reorganized table of aesthetic-related records (see Figure 3.7) was used to determine whether there was interaction between the type of lesson and the phase of planning and implementation. From this analysis, the researcher was able to make some assessment of the quantity and variable quality of the teacher's discourse about mathematical aesthetic within various temporal phases of lesson development, as well as among the lessons, themselves. The records were then sorted into the four levels of perspective ("Ag", "Agg", "Ag?", and "Agg?") within each lesson phase and grouped according to the researcher's own description of the actual pedagogical function that they served. Furthermore, ongoing data collection and iterative refinement of analysis prompted a renaming of the "Ag?" and "Agg?" categories. They were renamed "Emergent" and "Dynamic," respectively. It was found that within each of the "Ag?" episodes the teacher's perspective, though focused primarily on particular features and actions of individual student artifacts, changed to that of a much broader view of the whole group's artifact. The initial focus of the teacher dialogue was on the perceived movement of individual artifacts and then their coordination into a larger scale picture. In this sense, the whole group artifact was seen by the teacher to emerge from individually created artifacts or the actions thereof.

Conversely, within each "Agg?" episode the teacher referred mostly to the whole-class generated artifact as a unified picture, comprised of moving parts. The teacher made reference to an individual student's object only as parts of the whole, contributing to (or distracting from) the larger scale, class generated picture. Within each "Agg?" episode, the teacher emphasized the whole-group view of the artifacts and then shifted to the individual vantage point for the purpose of as a means of assessment. Thus, this perspective was renamed "Dynamic." From the stand point of coding, the difference between "Emergent" and "Dynamic" perspectives is in the particular level of perspective that gets referenced first and more often within the given episode. Chapter 4, section 4.2 gives a thorough and more detailed description of all four levels of perspective, Individual, Group, Dynamic, and Emergent.

The table in Figure 3.7 shows that for each level of perspective, the researcher assigned one or more associated pedagogical functions (numbered, column three), which served as a way to consolidate multiple records (columns four and five). The definition of these pedagogical functions was taken from the Concept and/or Description columns from the original database table (depicted in Figure 3.6). These provided a way for the researcher to ascertain the purpose for which the teacher made particular utterances and actions at given times or for specific lessons. Figure 3.7 represents only a partial listing of records from the table presented in Appendix B.

Rule for Points/Conception			**U04011501**	**U03112001**
	Agent	1. Space-creating function; Defining, constraining, and delimiting the space for agent-level activity.		15:15
		2. Aesthetic appeal of diverse agents.	33:35	
	Aggregate	Aesthetic appeal of the image of the aggregate.	31:00	4:56; 19:52
	Dynamic	Teacher perceives the aggregate as a composite structure of moving (movable) parts. Aggregate is perceived as a collection of agents. Aggregate is seen to embody the one commonality of a collection of diverse objects.	33:40; 34:18	5:36
	Emergent	Aggregate is perceived as an emerging pattern of agent behavior	34:39	5:36; 7:24; 18:25
Rule for Points/Planning			**U04011301**	**U03112001**
	Agent	1. Space-creating functions	2:30; 3:13; 6:45; 6:59	8:00
		2. Predicting potential caveats in the agent activity. Potential student difficulty with the agent-level activity. Navigating the contours of the space for agent activity.	7:06	7:30
	Aggregate	1. Predicting the aggregate image; Teacher asks students to predict beyond the aggregate artifact of the present lesson	2:20; 7:35 9:40	
		2. Aesthetic/ passionate for the aggregate.	6:35; 7:50	
	Emergent		7:24; 10:40	
Rule for Points/Implementation			**U04011603**	**U04011601**
	Agent	1. Delimit space; Gives the rules that define the boundaries of the space for student agent-level activity; gives counter-examples; wrong answers. Expand space; Broadens the space of activity by increasing the number of possibilities for viable student responses; explains the possibilities and features of the space; give examples and counter-examples; Give hints	26:20; 28:57; 38:36	16:45; 21:00; 33:00; 40:56 39:42; 39:54; 40:56; 44:23; 1:04:40; 39:42
		2. Aesthetic/ passion for diverse agents.	32:58; 39:29	
	Aggregate	1. Predicting the aggregate image; Teacher has students predict the final aggregate image. Teacher asks students to make inferences beyond scope of the present activity.		47:31; 48:50

Figure 3.7 Records sorted by lesson, phase and level of perspective with descriptions of pedagogical function.

Data analysis was initiated simultaneously with data collection, as is generally practiced in case study research (Stake, 1995; Creswell, 1998). The coding procedures for analysis were used to make sense of Martha's ways of discussing the lessons during planning and implementation. The interpretations of the data were verified with colleagues, mathematics and science educators from the Teaching and Learning Department at a local university. Also, since Martha and the researcher had collaborated on other projects, the case study simply became a part of an on-going dialogue about mathematics and pedagogy. With such an informal setting, she was able read and comment on several of the preliminary drafts of the analysis and interpretation of the data. Perhaps, because of this ongoing, close collaborative effort between the teacher and researcher, she gave no suggestions for changes.

CHAPTER FOUR

CASE ANALYSIS

Enacting Generative Design with HubNet

Martha seemed genuinely excited to use a classroom calculator network in her mathematics classes. She had spent considerable time with the Participatory Simulations research team, developing and creating activities to try in her classroom. She even found the bravery to run them in class herself, in the midst of wandering researchers and graduate students from the local university. The network, still in its prototype stage of development, crashed so often that she could only run half (or less) of the activities she planned for each class period. As frustrating as that was, she still expressed hope in the technology to allow the students meaningful mathematical experiences. She seemed to appreciate most its ability to project visual images of the mathematics. In one exchange with her while planning a lesson on linear equations, she described the whole-group artifact as much in terms of aesthetic as accuracy and validity.

Collaboratively she and researcher practiced the lessons that she created, each taking four or five calculators to act as separate students in the participatory simulation. These activities were often based on approaches in *Function-based Algebra* (Stroup, 1997), developed in other contexts and modified to be used with HubNet. One such task was to create linear equations, all with one particular *x*-intercept of her choosing. In this case the point was (3,0), through which all of the lines had to pass as shown in Figure 4.2. Upon seeing the image of the completed task, she exclaimed, “That’s beautiful!” Caught up in the excitement of the moment the researcher chimed in, “There it is. That’s a star [that we were both expecting].” She continued with a big smile, “That’s simply beautiful. They’ll figure it out. I’ll just give them a few more hints.”

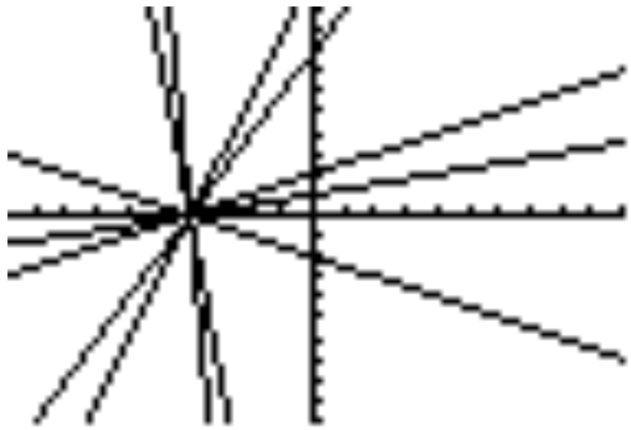


Figure 4.1 Calculator view screen, showing the likeness of the group generated artifact in the *Linear Family* lesson.

Wanting to support a reflective perspective, the researcher stepped back from the elation of the accomplishment and in a serious tone asked, “Why is that beautiful? Why do we say ‘beautiful’ when we see something like that?” “Well,” she thought, caught off-guard, “I think it’s creating a pattern, I guess. I can see all those lines,” she exclaimed, making a starburst gesture with her hands, “filling in and making one of those fine geometric beginnings of a mandala or something. I don’t know.” (Tape No. U04011301, 52:00)

Martha was surprised by the researcher’s question about beauty, in part, because she was initially unaware that one focus of the investigation was her aesthetic perceptions of mathematics. In fact, the researcher was unsure of what these planning sessions with Martha would reveal, relative to the teacher’s role in enacting HubNet generativity. The original conjecture of the investigation was that HubNet activities would create moments of pedagogical uncertainty for the teacher and thereby, elicit notions of mathematical proof. However, early on, notions of mathematical beauty began to make up a large part of the data, particularly in the discussions about the ways she intended to use the artifacts in class. Aesthetic perception of mathematics was a much more salient topic in our dialogue than proof. Pursuant to the methodological commitments of this investigation, detailed in Chapter Three, in which design research entails iterative refinements analysis (Brown, 1992; Collins, 1992), aesthetic evolved into the primary focus of the study.

The above episode, while getting to the heart of the matter in the present investigation, illustrates the difficulty Martha had in articulating the specific source of her enthusiasm for the pedagogical artifacts produced by the lessons. Her surprise in being asked about mathematical beauty may be indicative of the relative novelty of aesthetic as a meaningful concern in the mathematics education community (Sinclair & Watson, 2001; Wang, 2001; Foshay, 1991). A primary issue in this investigation is the manner by which this engagement with mathematical aesthetic might be described. Initially, the researcher and Martha lacked sufficient vocabulary to converse about mathematical aesthetic in ways that were meaningful in the context of teaching practice. The findings in this chapter are presented to illustrate the evolution of this dialogue about mathematical aesthetic by capturing the consistently reoccurring patterns in the teacher's utterances and behaviors, relative to her perception of mathematical beauty.

The lengthy dialogues and detailed vignettes are presented here to give the reader a thorough and accurate account of the ways in which aesthetic perceptions of mathematics emerged naturally as a part of the dialogue in both the preparation and implementation of HubNet lessons. Data were not responses to questions from already prepared interview protocols, but are discussions, not prescribed in advanced by the researcher, about the aesthetic nature of the mathematical ideas embodied in the actual or potential network-generated artifacts, students' engagement, and classroom discourse. The stories are organized into themes to show that mathematical aesthetic, as perceived by the teacher, both defined the boundaries for student activity engagement and shaped the mathematical discourse in and out of the classroom.

The episodes presented in this chapter show how the teacher's aesthetic perceptions were activated for pedagogical purposes, vis-à-vis instructional preparation and classroom discussions. The vignettes herein have been specifically arranged in order to explicate the nature of the teacher's aesthetic perception of mathematics, as it was dictated by (1) phases in the implementation process of network-supported lessons (conception, planning, and implementation) and (2) levels of perspective from which network-generated artifacts were viewed by the teacher (individual, group, dynamic, and emergent). These lay the groundwork for an overall finding, presented in the next chapter, that there exists an interaction between notions of mathematical aesthetic and generative design, as instantiated by network-supported activities.

Aesthetic in Phases of Generative Design and Implementation

As discussed in Chapter Three, a temporal structure existed for creating and teaching the HubNet lessons, which included a conception phase, a planning phase, and an implementation phase. The analysis of each of the episodes during pedagogical development of the network lessons revealed that the notion of aesthetic was ubiquitous throughout the entire generative teaching process. In the sections that follow, detailed vignettes are offered as illustrations of the primary pedagogical functions of aesthetic and its various instantiations in each phase of lesson development and implementation of network-supported generativity.

Conception Phase

From the outset, Martha already had affect in mind as a primary goal for these HubNet lessons, even during the conception phase. One day Martha, the researcher and another mathematics teacher on staff at the school sat down to work through Martha's preliminary ideas for a network activity, and we started a discussion of what her IMP I class had been working on. She described how they had been looking at geometric ideas and how to use sequences to find patterns and from patterns to algebraic formulas. (The IMP I curriculum was approximately equivalent to the traditional Pre-Algebra course, in terms of content.) Martha stood up and paced around near the white board explaining her previous class activities. The researcher and the other mathematics teacher on staff (there to find out the details of the research project) sat down at a table in the front of the room. Martha explained how she had been trying to get her students to come up with a rule for calculating the sum of the angles in an arbitrary regular polygon. She relayed to the researcher how she had used a table of values to help her students learn. "When I had my table out," she said, approaching to the whiteboard and pointing to a table of values, "I wrote this [drawing the polygons on the board] off to the side. They could see that but they couldn't quite come up with a rule." She explained the exercise was not necessarily geometry or algebra but a mixture. "It's more about, 'let's look at mathematics a whole different way.' Math is different than [you thought]." she explained. She later described how she wanted her students to use visual patterns and graphs "to start to look at math differently." (Tape No. U03111801, 11:00)

In another episode Martha and the researcher turned more specifically to creating a lesson using network activities. She wanted to use the network to create visual artifacts of the geometric patterns. She visualized the final artifact of the activity. In her conceptualization of the topic for the *Rule for Points* lesson, Martha said, "Mainly, what I want to do is introduce them to the coordinate system. So maybe we do our thing where we go in a say, 'everybody go where your x is positive and your y is positive. [Predicting hers and the students' responses.] Oh my gosh! Where did we all end up? In the first quadrant." She proposed further, "And then I thought we could go where x equals y. [Again, predicting the class response] Oh how interesting! It makes a line." Martha's comments suggested that part of the objectives of the visual projections was to arouse an affective response from the students. She wanted, through the aggregate images, to draw out an emotional response from the students that would indicate their appreciation for the qualities of the mathematical ideas.

Martha then explained further how she wanted to use some of the geometric pattern activities that the class had been working on. She had a list of analytic functions (rules) that the class had already discovered, and she wanted each student to pick an x-value and move it to the correct location on the coordinate system, according to the rule. She proposed as if talking to the class, "Let's see what happens if we all use a different x and find our y using our rule." In this initial stage of lesson development of Martha's conception of what the final visual artifact could be about pedagogically consumed most of the discussion between her and the researcher. She explained her objective, “ I want them to see that...wow! With these rules when they graph, make something [a visible pattern].” (Tape No. U04011301, 2:30)

The above episode supports the claim that affect was one of Martha’s primary objectives. For her, having the students appreciate the wonder of mathematical constructs was a stated goal. Use of phrases like “wow” denotes some expectation of surprise in the students’ experience of the mathematics. Furthermore, statements like “seeing math differently” during the conception phase of lesson development convey a sense of surprise and were recorded (by the researcher) as indicative of aesthetic appreciation of mathematical constructs. Martha wanted to have a visible display of mathematical aesthetic by making use of whole-class generativity. The whole group image that would be projected to the front of the classroom was seen (by the teacher) to embody aesthetic characteristics in the mathematics, and she planned to make this point explicit in the classroom dialogue, as illustrated further in the following vignettes from the planning phase.

Planning Phase

The planning phase amounted to Martha and the researcher practicing the role of the students on the calculators. Sometimes this meant that each one of them would operate several calculators (four or five) at once; Most of the time spent during the planning phase was on predicting the ways the students would interact with the lesson and on creating appropriate activities for them that fit their level of understanding. In the planning discussion of the *Linear Family* lesson (described in more detail at the beginning of this chapter), Martha and the researcher we were concerned mostly with whether the students were knowledgeable enough to create linear equations, if they were given only one point, (3,0). They had the following exchange.

MARTHA: So they would be thinking, 'I need to come up with some equation that if I put in three, I get a zero. They could say 'I could times it by zero' or they could say x minus three.'

RESEARCHER: So x minus three would work.

MARTHA: Sure. If x is three, y is zero.

RESEARCHER: Ok, so there is that equation [pointing to the calculator].

MARTHA: So then we'll say draw the line on your paper. But I'm wondering if we want to limit them from using x equals three.

RESEARCHER: I don't know.

MARTHA: Well, some of them might try that. That's another way to get from three to zero. And some of them might just times it [the x-value] by zero [resulting in the equation, $y = 0$], especially since we've just been using the zero property.

RESEARCHER: But is this [pointing to the calculator at $y = x$-3] the only line they could come up with?

MARTHA: No. Some of them will times x by something. You could change the slope and play with it for a while. Some people will try to multiply the whole thing [x-3] by something. But we haven't played with this for a while. Some of them might times it by zero...

RESEARCHER: And that would give [making a horizontal gesture with my hand] this line.

MARTHA: ...Which would go through the point. But we'd need to force them to give us diagonal lines instead of [horizontal] ones.

After pausing for a while, Martha continued, "Well, some of them might try to multiply [*x*] and see what they have to add [or subtract] to make it work. And by then they'll all have something they know how to do. And so what's going to happen is I'm going to get all these different lines." They continued their dialogue.

MARTHA: What I think they'd want to do is try things and test them on the calculator for a while and see if they could get one that goes through the point. I mean, I could give them some hints, but maybe I'll be happily surprised. There's going to be a few that are maybe, you know...I have some kids in this class [IMP III (Algebra II)] that I don't think anyone thought they would learn anything past geometry.

RESEARCHER: Oh, really?

MARTHA: This may be a little out of their comfort zone. But I'll say, 'I want you to try to see if you can make one and see if you can make another one. And when they do we can ask them. I'm assuming that some of them will say, 'you can change the slope [of the line, $y = x$-3]. Then I can say, 'well, do that and then try it and see if it does work. I don't want to tell them, I want them to try it and see if it works or not. They'll try different things. It shouldn't be too confusing. They should notice that if you change the slant, then it won't go through there [pointing to negative three on the calculator].

RESEARCHER: Negative three.

MARTHA: Because to stay there [pointing to the point (3.0)] this [pointing to the *y*-intercept] has to change if your slope changes.

RESEARCHER: The question is 'how much?'

She immediately began to figure out a strategy to determine the change in the *y*-intercept, if one were to change the slope. Working on scratch paper, she tested several different cases where she changed the coefficient of *x* and added or subtracted a value to keep the solution (3,0). Afterward, she said, "Some of them will approach it algebraically, by multiplying the whole thing by something or they might try to add or subtract something [from the *x*-term]. They will approach it in many different ways, and it might be a struggle because of the way they approach it. But I'm willing to let them explore and try things, because even if you're way off, when someone explains it to you it makes a lot more sense." (Tape No. U03112001; 30:00)

As she planned for the generative lesson, she demonstrated an understanding of mathematics as a field of inquiry. She did not, as the researcher had expected, evaluate the different solutions in terms of their pedagogical benefits, i.e., the mathematical ideas emphasized by various solutions. Instead, the teacher seemed mostly concerned that all the students had the opportunity and ability to offer at least one solution.

Martha's pedagogical concerns seemed to be articulated separately from her perception of aesthetic in either the whole-group or individual artifact. She was concerned with students' ability to complete the task and make a contribution to the whole group artifact, but much less with creativity or qualitative assessment of their generated constructs. However, she herself explored the capability of the task to foster or allow inventiveness of the students. Martha wanted a task that would be open to all of her students and give them the chance to explore. She planned for students to "try it [student-generated construct] and see if it works."

Even with this apparent separation, emphasis is still placed on the appearance of group artifact, as in Martha's explanation of finding a line containing a given *x*-intercept. In her conjecture, "notice if you change the slant, then it won't go through there [the given *x*-intercept]," the image remains the embodiment of the mathematics and it directs mathematical inquiry process. However, it may not necessarily be aesthetic perception that guides, but simply a notion of the "correct" form of the image. Still, it is the image, itself, that mediated the discourse and opened the possibility of aesthetic concerns to be thought about simultaneously with pedagogical ones. Would all viable individual constructs be represented in the group-constructed artifact? Would students be able to identify and distinguish the various classes of individual constructs? Would they recognize and appreciate the unifying theme of the seemingly disparate objects?

It was this unifying theme in the group constructed artifact of a linear family (Refer the Figure 4.1), as was illustrated in the planning episode given at the beginning of this chapter, that was the source of the teacher's aesthetic appreciation—that many unique and separate student-generated solutions could be perceived as a unified and readily identifiable pattern. In observing the whole group artifact, Martha could "see all those lines filling in and making one of those fine geometric beginnings of a mandala." This kind of focus on the whole group artifact, refining it to match what was in Martha's mind, may explain the larger number of records associated with whole group aesthetic in the planning phase, as opposed to the larger number of records for individual that accompany the conception and implementation phases. Not having real students around to express their own mathematical inventiveness, there was no reason for Martha to focus on the individual constructs, but rather that of the group. Table 4.1 shows that there are three times as many whole group aesthetic related records as individual in the planning phase, while the conception phase has half as many group as individual. In the implementation stage the numbers of both group and individual referenced statements by the teacher were quite high.

Table 4.1
Number of aesthetic-related records found in each phase of lesson development.

	Individual	**Group**	**Total**
Conception	24	13	37
Planning	8	24	32
Implementation	37	47	84

The group generated artifact is where most of Martha's aesthetic appreciation for mathematics could be seen, as was the case with the *Product of lines* lesson planning, in which Martha and the researcher practiced the participatory simulation on the calculator network. Again, they tried to simulate the network activity for a class of students; so each of them operated four calculators and completed the participatory simulations activity as eight separate participants, resulting in the group generated artifact depicted in Figure 4.2. The following exchange between Martha and the researcher demonstrates how she planned for the discussion:

MARTHA: I wanted to point out a few of the similarities in the graphs [between the graph of the parabola and the lines].

RESEARCHER: What similarities did you want to point out? What kind of answers are you looking for?

MARTHA: Similarities would be...Their going to be looking at theirs a lot and their paper;

so similarities would have something to do with obviously where they go through the roots, the x intercepts. And that maybe the slopes have a relationship to the vertex. I don't know if they're going to have enough data to collect that information.

RESEARCHER: What connections do you want them to see? That is my question.

MARTHA: Well I like all of those. Well the linear equations that make the quadratic go through the roots of the quadratic. So they cross the x-axis at the same points. The quadratic is the product of the two lines. And I hope some of them will see that the less steep the slope of the two lines, the wider the quadratic is.

RESEARCHER: So the lines relate to the [interrupted]?

MARTHA: ...To the spread? You call that spread?

RESEARCHER: Well, we've been calling it fatness [laugh].

MARTHA: Yeah. I don't want to use fatness. This is a sensitive area. Let's not do 'fatness.' The coefficients (slope) in the two linear equations would affect whether it was inverted or not. They'll either notice it or not and if they don't I might just have to say, "Well, does it make a difference? Or like did anyone have one of their linear equations have a negative slope?"

RESEARCHER: I wonder if there is any value in not multiplying the two (linear equations) together.

MARTHA: Yeah. I decided that already, mainly because, I thought, well, practice is fine, but I don't want it to eat up our time.

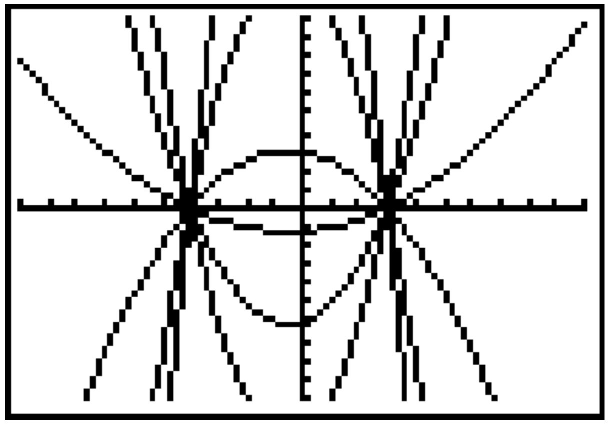


Figure 4.2 Reproduction of the image of the whole group generated artifact in the *Quadratic Family* lesson.

The projected macro image was seen to embody the many varieties of possibilities of a problem solution or cases. Martha saw the upfront projection of the family of parabolas as a static image, depicting some aesthetically appealing mathematics construct. Accordingly, Martha went as far as predicting students' emotional responses to the aggregate image, as if affect were a high priority. "We will get some definite 'cools' from this," she commented. "We'll just let them talk about it, and they will go 'cool.' Someone will." It was the picture that made all the difference to Martha, because it displayed all of that variety in a unified image, which was "beautiful" to her. (Tape No. U04011301, 1:20:00)

The above episodes from Martha and the researcher's planning sessions illustrate how aesthetic was overtly stated as a goal of instruction. Martha planned encourage an affective response through the visual artifacts. Getting to the final whole group artifact was a top priority, not "wasting time" on long algebraic procedures like binomial multiplication. Moreover, the following episode demonstrates how the pedagogical artifacts of generativity were useful as tools for mathematical exploration. In the preceding case, Martha and the researcher spent considerable time during this episode, trying to understand how the two lines (linear factors of the quadratic) related to the position of the vertex of their respective parabola. The visual image of the family of quadratics, along with the linear equations that made them, gave both Martha and the researcher an opportunity to learn mathematics new to both of them. The projected image of the generative artifact allowed Martha the opportunity to point out to her students the connections between different mathematics constructs, exploring ideas that were new to her. She was willing to leave plenty of room for unintended mathematical ideas and discoveries to emerge from the class discussion. For Martha, the surprise of mathematical novelty was aesthetically appealing.

Implementation Phase

While aesthetic seemed to operate sometimes in the background of the teacher's thinking about the mathematics as Martha and the researcher planned the activities, more often, as Table 4.1 shows, it appeared more overtly in the teacher-led classroom discourse. The total number of aesthetic references during the implementation phase was well over that of the previous two phases combined, 84 for the implementation phase as opposed to 37 and 32 for conception and planning, respectively. This may have occurred because the teacher had the opportunity to see aesthetic in individual student-created artifacts in addition to those of the group during implementation. On the other hand, the conceptualization and planning phases, Martha could only perceive the whole group artifact.

Some of the evidence presented in the following episodes suggests that the HubNet technology itself is novel enough to the classroom context to engender student and teacher excitement. This could possibly have confounding effects on the interpretation of the data, as one might plausibly infer from these observations that students were simply excited about using something new. The students and teacher's excitement may not necessarily be an affective response to the images as mathematical constructs, but as novel objects to be explored.

Martha implemented the *Rule for Points* lesson in her IMP 1 class, made up of seventh and eighth graders. The students were so excited to use the new "toys" on their desks. Each one had a graphing calculator with a wire coming out of it, connecting it to three other calculators via a hub. Despite the technical difficulties, they had all already logged on to the network activity, *Function Activity*, and begun to play around with it. On each student's calculator view screen there appeared one randomly placed point in an *x-y* plane. For each student calculator, there appeared on the overhead screen in front of the class, a corresponding icon (e.g. car, square, circle, triangle) positioned in the exact same location as the point on the calculator. Each student immediately identified his own corresponding icon on the upfront screen and began to use the arrow keys on the calculator to move it around.

There was noticeable excitement in the class as students raced their points on the upfront screen, chased each other around the screen, or tried to occupy the same position as a classmate and cover over his icon with their own. In a sense, one might think of this as an unbounded space for exploration. But it was obviously unsettling to Martha as she tried to keep the class on task.

"Put that calculator down, Dawn, or I am going to have to take it away from you," she said calmly, in obvious frustration. After scolding a couple of other students for the same thing, she said, "We're going to talk a minute, and you will get to play again. I promise."

Once she settled the class down, she continued by issuing a challenge to the entire class. "What I want you to do is I want you to move to a place on the screen where your x and y are positive." Now instead of having free exploration of the entire space, Martha's challenge confined the student activity to the first quadrant, and although there are an infinite number of locations in which to position individual points, the space for student activity was still constrained much more than before.

According to the teacher, in this class many of the students either had a very superficial understanding or were just learning about the Cartesian coordinate system for the first time. Pointing to the white board where she had written "x=1" and "y=3," she explained how the numbers could be written as the ordered pair (1,3). She asked, "How many of you have played [the game] *Battleship*?" Several students raised their hands. "It's kind of like *Battleship*. It's giving you coordinates. So if I were to plot this on my graph [pointing to a coordinate system she had drawn on the board], I'd go over one for my x and up one, two, three for my y. So you guys put your point on your graph [on a handout]. Then we'll play with the machine." They students were studying a geometric pattern, the number of adjacent triangles formed by adding new line segments. Martha had asked them to choose one solution and write it as an ordered pair. Once the students plotted their points on paper by hand, they all logged back on to the network to participate in the *Rule for Points* lesson.

As the students logged on, they immediately began to move around the screen up front. There was considerable excitement in the class. Martha went through a few of the exercises that she had planned for them to get familiar with the system, like moving their points to different quadrants and so on. In the end, Martha used the network activity as an extension of the geometry activity from the IMP I curriculum (mentioned earlier). “Remember your point that you picked for your triangle and match sticks?” She asked the class. “Remember how we all chose a point? I want you to move your [icon] to your point that you have on your paper.” The class was still buzzing with excitement as they moved about the screen. Seconds later several students shouted, “Done!” as if they had been racing to finish. “Ok. Well, let’s give other people a minute. They might have been further away from their point,” Martha responded.

As the rest of the class completed the exercise, several students complained that their icon had been covered up on the screen up front. “I covered up someone,” called out one student. “I covered up someone, too,” said another. Because students’ move around their individual calculator screens, which corresponds to the icon on the up front display, some students attempted to occupy the same point (location). However, the limitations of the NetLogo display preclude more than one icon for a particular location. “Hey! What happened to me,” another student complained. “What’s your point [coordinate]?” someone asked. “Two, five,” he responded. “That’s mine, too,” she giggled. “I just covered you up.”

Martha asked several of the students, who had developed their own rule for the geometric pattern, “Who had a rule? [Pointing to one student] You had a rule, didn’t you? Would you put in a negative number?" The class continued working for several minutes. On the screen up front a line began to form in the first quadrant, with two or three errant points. Martha said, "Well, it looks like most people are starting to make some kind of interesting pattern, but not everybody." (Tape No. U04011604, 28:00)

Despite the obvious novelty, there is ample evidence that some of the mathematical discourse in the classroom during the implementation of the network to lend an interpretation of the existence of aesthetic perceptions. Martha's continual insistence on emphasizing what she thought of as "interesting" about the whole group artifact during planning lends credibility to this interpretation.

Sometimes the network allowed students a chance to explore mathematical ideas in ways that the teacher had not anticipated doing either the conception or planning phase lead to student motivation to impress the teacher with creative mathematical thinking. One story that clearly illustrates this unplanned student engagement occurs in the class in which Martha implemented the *Linear Family* lesson. It began with Martha sending to the network and consequently, to each student's calculator a point on the x-axis, (3,0). "Now," she orders, "I want you to write an equation that goes through that dot. This will be easy for some of you, but some of you will be like 'Oh, I haven't done this in a while. How do I do it.' But make some equation, y equals something x...do something to x so that it goes through that dot." Some students got to work on the problem immediately, but others seemed lost by the directions. So Martha gave hints to the class based on their previous work in the IMP II curriculum (equivalent to Algebra 1 and Geometry in the traditional mathematics curricular sequence) on in-out tables. She turned to the white board and began to discuss, "Maybe you have an in and out table that has an x and a y on it. With your x [value] three and your y [value] zero, give me some rule that makes that true." One student called out, unsure of himself, "Oh, um, y equals x minus 3?" Martha responded, "Ok, there is our most simple one probably." She tested the point in the equation, substituting three for x and calculating the y value of zero. "So it works," she said. "So next time you might want to start with y equals something times x and figure out what you would have to add or take away to make it equal zero, something more interesting. Some of you might want to put a $3x$ here. Then what am I going to have to add to be able to go through this point. You could pick a one; you could pick a four; you could pick a two. I have some that I'm going to force some of you to pick." She walked around the class for several minutes, helping some students complete the activity. The students were then asked to send to the network a linear equation whose graph passed through the given point.

She discovered afterwards that some students had not created linear equations. One student was adamant that not all of the equations that went through the point would be "straight lines," as he had already created a non-linear curve. Martha then revised her initial directions, "For what I want to do right now, I want two linear equations. I didn't specify, but I'd like a linear equation." She conceded to students that she had not expected such creativity in thinking "outside the box" about the problem. After working through the activity another time and about 41 minutes later, the whole group artifact was projected on the up front screen. It represents the pedagogical artifact of Martha's lesson in which she asked the students to create a linear equation, whose graph passed through the point (-4,0). Refer to Figure 4.3. (Tape No. U04011602, 18:00)

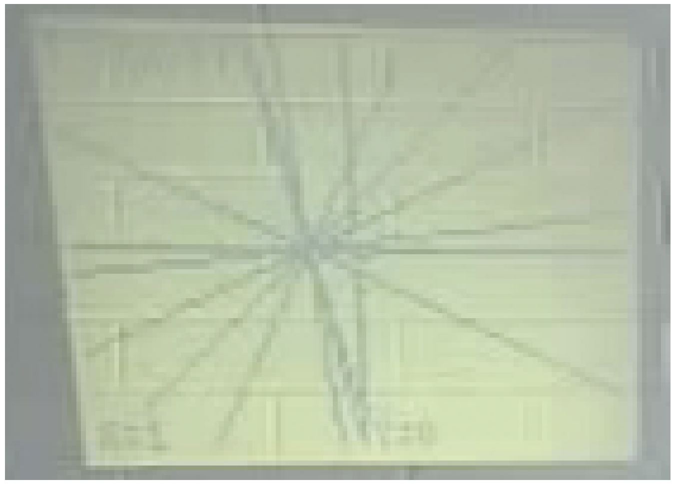


Figure 4.3 Still image from a video clip of the calculator screen appearing on the up-front display during the *Linear Family* activity.

The technology allowed for student creativity and inventiveness to such an extent that it was surprising to Martha. She had to retroactively redefine the rules of the activity so that students would be limited in their choices of functions to send. However, it is not clear that this incident engage the same perception of mathematical aesthetic to the same extent as that of other surprising student generated constructs. Still, Martha seemed somewhat pleased with students' inventiveness in her concession that she had made a mistake in not restricting them to submit only linear functions. This is evidence, for the first time in this investigation, of the capacity of the technology to support student creativity and inventiveness, which opens the door for aesthetic of individually created constructs, not just the group artifact.

The teacher's use of language during each phase of implementation of HubNet lessons evidenced an aesthetic appreciation of the class-generated artifacts, both whole group and individual. Aesthetic perception played a prominent role in her teaching throughout, from planning for affective responses to the appearance of whole group artifacts on the up front display in conception and planning phases to compelling students to generated "more interesting" artifacts during implementation. On several occasions Martha described the whole group artifacts as beautiful, representing the variety of disparate objects forming a unified design.

In the following section, episodes are organized according to the level of perspective of the teacher (or students), from which the perception of mathematical aesthetic arises. Although the emphasis in the section will be the perspective, phase in the lesson development will continue to be cited within the vignettes to give the reader additional reference and credibility to claims made in this section that aesthetic perceptions of the teacher are apparent in all phases of generative lesson design with HubNet.

Aesthetic in Levels of Perspective

While it was anticipated that HubNet-supported generativity would make aesthetic visible vis-à-vis pedagogical artifacts, what was not clear was the manner by which the teacher would enlist aesthetic as part of her pedagogical decision-making process. The following episodes are organized according to four different levels of perspective from which the teacher's aesthetic perceptions were found to be operative in network-supported generative activity design. These levels of perspective included what are called individual, group (aggregate), dynamic, and emergent perspectives. These levels were seen by the researcher to initiate three different types of aesthetic perceptions from the teacher—static, the perception of the whole group artifact as a single constructs, comprised from various and individual artifacts, as representing special or unique cases; dynamic, the notion that the whole group artifact is a composition of smaller moving parts; and emergent, the idea that the whole group artifact as emerging from the actions of individually created artifacts.

Static aesthetic within group and individual levels of perspective

Static aesthetic operated from the whole group level as well as the individual level of perspective. It is characterized by the teacher seeing the whole group generated artifact as unifying an entire space of infinite possibilities of individual artifacts around a single mathematical construct. Some individual artifacts were seemingly more preferred by the teacher than others, and were thus, deemed aesthetically pleasing in their own right.

The vignettes given below will emphasize the ways in which the teacher's perception of mathematical aesthetic was engaged by her viewing the network-generated artifacts as static constructs, perceived on one of two levels, whole group or individual, separately. "Static" is meant to imply that the level of perspective from which the projected artifacts are viewed is static. There is no shifting between whole group and individual perspectives in perceiving the mathematical object. On the other hand, there is the sense in which the aesthetic of the artifacts as seen by the teacher can also be considered as static in nature. The artifact is perceived the teacher as a static image, beautiful for its patterns, symmetry, or uniqueness.

The group level of perspective is defined as Martha's perceiving of the appearance of the aggregate image generated by the entire class. It is a fixation on the "big picture" and not necessarily the individual student generated artifacts within that picture. Because the image of the entire class' activity would be projected to a screen upfront for all to see, Martha would seize the opportunity to use it as a pedagogical tool. As she and the researcher worked through the pre-scripted exercises, they talked about how students might perceive the image. It was in the process of planning for the Quadratic Family lesson that aesthetic seemed to come into the conversation more than in any other episode. The following episode from the planning session of the Quadratic Family lesson illustrates the static perception of aesthetic from the group level of perspective. In this case, the whole group artifact is generated by each student in the class sending the product of his chosen two linear factors to the network; thus, creating a variety of different quadratics. The unifying theme, moreover, is the shared pair of roots (x-intercepts) by an infinite variety of quadratics.

In this context, the teacher described specific features that she wanted the projected image to have in order to convey the "big idea" that she had planned for the students to understand. The first challenge that the students would be given for this particular exercise was finding an equation of a line that passes through a fixed point. Standing in as students, Martha and the researcher sent different varieties of linear functions through the point. Afterwards, they sent to the network their products of various linear factors, thus, constructing the family of quadratics. Refer to Figure 4.2.

As the whole group image began to appear on the calculator display, Martha exclaimed, "Oh, that's beautiful! Look at that, ooh! But it is pretty impressive when you see it go right through the spot with the other guys. That's pretty impressive." The researcher had not yet viewed it on my calculator and she said, impatiently, "You've got to see these equations. It looks pretty cool." You mean with the lines?" asked the researcher. "They'll see that. They'll see that [the image of the family of parabolas]," she replied. "Will they see that?" The researcher queried further, unsure whether the flow of the activity would make the screen too busy for the students to see what she wanted them to see. She assured, "They will when we resend, because everybody's line is going to go away, I think. But it'll still be on their calculator until they say that they're done and resend it."

After the teacher and researcher had created the whole group artifact together, they both shared our concerns that students may not be capable of sending to the network quadratics with non-integer leading coefficients. In fact, Martha wanted to make sure that some of the students created "half ones" [lines with fractional slopes that would, when multiplied together, yield quadratics with fractional leading coefficients, and thus, appeared more compressed and flatter than other quadratics]. She was adamant that the projected whole group artifact had to have quadratics with fractional leading coefficients—so much so, in fact, that she planned to compel some students (the more capable ones) to choose predetermined linear equations with non-unitary slopes. This would ensure that the image had a variety of quadratics with the same two roots, thus, representing a quadratic family of infinite order. For reasons that were motivated by aesthetic concerns, Martha wanted to insure that a variety of quadratics be represented in the projected image, including as many differently sloped lines as possible. So important was her aesthetic sense of the activity that Martha even went as far as appointing certain people to send specific constructs to the projected image to ensure that all of the variety was represented in the pedagogical artifact. The following exchange was typical of our planning sessions, where the teacher would ensure that that artifact would have all of the different parabolas she wanted.

RESEARCHER: No one's going to pick half.

MARTHA: I'll pick a couple. I'll say, "I want you to do this to the x.

RESEARCHER: Multiply the x?

MARTHA: I'm going to say, "Dustin, I want your equation to start like this: negative x divided by two. And then you figure out what you need to do...Figure out what you need to make that. So I'll pick a couple of people and go will you make an equation that has x look like this and see if you can get it to go through. If someone wants an extra challenge, I'll see if I can get Anthony to do x divided by three.

RESEARCHER: That might be a little bit easier, especially through (3,0).

MARTHA: So this one I'll divide by two. Can I divide it by more than two? They can do it. These are algebra students. Some of them would freak out because they're like "Oh my Gosh, how do I do it? Ezra and Ruth and Danny they can pull this stuff off with no problem. "Can you try to make an equation that is y equals x divided by two or three or something?"

RESEARCHER: You know what? I have a feeling they're going to get good at manipulative stuff.

(Tape No. U0401130, 1:04:00)

As Martha and the researcher continued to practice the *Quadratic Family* lesson, she explained how she planned to make sure that as many varieties of parabolas would be present in the final projected image as possible. She was adamant that the image should consist of inverted parabolas as well as those of varying vertices and amplitudes. Martha wanted the image to reflect the broad range of quadratics in this family. The sense of mathematical beauty comes from the fact that the variety of student-generated artifacts can be condensed into a single mathematics construct, embodied in a whole group-constructed image.

The teacher's vision of what the final aggregate image of the simulation should look like seemed to guide her thinking about the goal of the entire activity. In this sense, aesthetic seemed to be used as an assessment measure revealed as an urge to fill up a perceived mathematical space, a representation of the infinite varieties. Her practice was guide by a sense that all varieties represented in the aggregate image, not necessarily every possible function in this family, but merely the sense that the symmetry of all the different possibilities were distinguishable in the projected group constructed image in order to give the impression of infinite variety. Was there a variety of inverted parabolas in the image, for example? Stretched ones? Compressed ones?

In this way, Martha used the aesthetic perception of the aggregate image as a way to check for saturation of the space. Not only were the generative artifacts appreciated for its aesthetic, but for the cognitive connections that could be made by having a visual representation of the students' thinking. The teacher pondered what students could learn about quadratics by having a family or class of quadratics. Some example that she came up with were that quadratics can be classified by their roots, or that given the roots of a quadratic, one can easily find other quadratics that satisfy the exact same solutions.

In addition to perceiving the group artifact as one fixed image, the teacher also focused on some of the individual student generated constructs within the larger one. It was found that the teacher's perception of mathematical aesthetic also arose from her seeing some individual constructs within the group-constructed artifact as aesthetically appealing for their uniqueness with respect to others. Individual level of perspective is characterized by the teacher focusing on an individual student's artifact without consideration of the larger group constructed image. Below is an episode that showed the teacher also perceived aesthetic in individually generated artifacts.

During the implementation of the Linear Family and Quadratic Family lessons, as the students worked to complete the task, Martha encouraged the students several times with comments like, "I'm seeing some interesting things up there." When students were asked a second time to send linear equations that went through the given point, she said, "Try to come up with another one. And let's try to use our more interesting one [to send to the network to be projected upfront], ok?" Furthermore, since the Quadratic Family (figure 4.3) lesson was an extension of the Linear Family lesson, she encouraged the students, saying, "How about now, we take our two most interesting linear equations [from before]...No, you don't get to use my boring ones, ok? Write down [as a product], not these two [pointing to her simple linear equations with unit coefficients], but your most interesting one [linear equation] for this one [referring to the first point (3,0)] and your most interesting one [linear equation] for that one [referring to the first point (-4,0)]. We're going to send this [product of the two] as our new equation. "Keep them in this form [the linear equations in slope-intercept form]. Don't put them in standard form, and then we're going to put this [the un-simplified product of two linear factors] in as our new equation."

The students worked for several minutes, some discussing their work with other students. Martha also walked around, helping. Martha tried to hurry the class along to get to the display of the final whole group artifact, shouting, “Everybody, let’s send our [equations]. “I’m going to wait until everybody sends them before I get [display] them.” She impatiently quarried the class, “Everybody sent them? That table...you guys sent yours? Ready? Ok, now I’m going to select and view them.” Figure 4.4 is a still image from the video clip from Martha’s class showing the pedagogical artifact in which students multiplied lines from linear families with roots at (3,0) and (-4,0). It represents a family of quadratic functions with simultaneous roots.

This was illustrated in the implementations of the Linear Family lesson. After all the students had sent their graphs to the network, they were projected onto the screen upfront. Martha began the discussion by asking, “So how many of you expected it to look like that [pointing to the overhead screen]?” One student responded, “I thought they were going to be like this [making a v-shaped gesture with his hands].” Martha pressed, “Why do you think it didn’t?” “Because they’re all positive,” the student replied. “Yeah and also because the easiest ones were positive. This is the first time you did it, so you thought ‘I’m just going to multiply it (the slope) by something positive rather than negative’, so we ended up with a lot of positive slopes.”

Martha’s class repeated the activity, and the resulting aggregate image looked more like Martha had anticipated. Upon seeing the new aggregate image, she exclaimed, “Oh my goodness! So tell me what’s the same about this one (referring to the overhead screen) than the one before.” One student commented, “There’s more negatives.” “Yeah. It seems like we have a few more negative slopes here,” Martha replied. “It’s more spread out," commented another student. Martha elaborated, "Yeah. It's looking a little more spread out. People are getting more comfortable. They're starting to use the negatives (negative sloped lines), and it makes an interesting design." (Tape No. U04011602, 32:00)

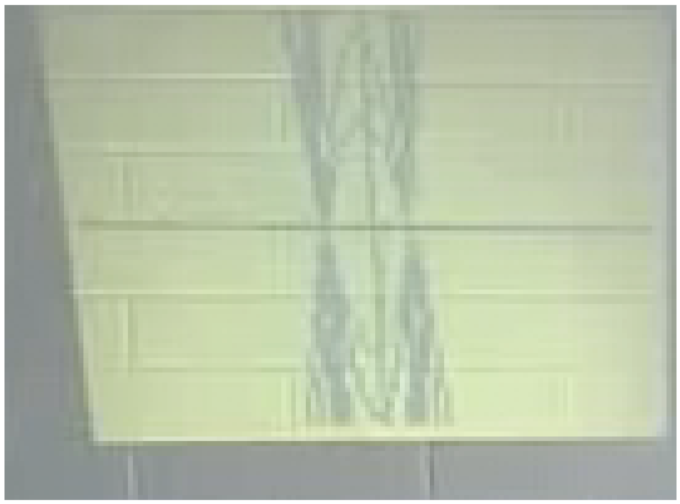


Figure 4.4 Still image from a video clip of the calculator screen appearing on the up-front display during the *Quadratic Family* activity.

This episode suggests that Martha placed special emphasis on interesting individual as they fit a larger design framework, which she had in mind. However, the individual were aesthetically pleasing by themselves for their uniqueness and rarity. She seemed to make impromptu evaluations of the mathematical aesthetic embodied in each one, denoting some as "simple ones" as opposed to others as "more interesting." Some student-generated artifacts were deemed more interesting because they were seen as less trivial cases of a particular mathematical construct. In those cases students were encouraged to "figure out" viable solutions to the problem, and that "figure[ing] out" is what made the constructs more mathematically interesting and therefore, aesthetically pleasing to Martha.

The teacher's perception of mathematical aesthetic acted as an instructional design constraint not only on the projected whole group images, but also individual student-constructed artifacts some of which were pointed to as more aesthetically pleasing than others. Individual perspective gave the teacher a way to evaluate each student-generated artifact one at a time. She could make comparisons of the "more interesting ones" and the more trivial, "boring ones", assessing the student creativity represented by and individual student generated artifact. The artifacts served as a proxy for student thinking and activity engagement, and Martha's perception of mathematical aesthetic was effectively a factor in pedagogical decisions she made in discussing the artifacts.

The preceding episode also illustrates obvious pedagogical drawbacks of an overbearing sense of mathematical beauty, relative to students' freedom of exploration. It was found, at least in one instance, that the aesthetic perceptions of the teacher possibly worked to limit students' creativity by the teacher's forcing them to choose predetermined responses to the lesson. For Martha some individually generated artifacts were such important components in this aesthetic for the group artifact that they had to be present, even if it meant detracting from students' agency in the activity and the development of their own sense of mathematical aesthetic. She wanted a completed image to match what was in her own mind. Other pedagogical goals were subjugated to an aesthetically motivated goal about the appearance of the final whole group artifact, limiting student inventiveness.

Dynamic perspective and aesthetic

While the individual and group level perspectives allowed a static expression of mathematical aesthetic, evidence showed that the teacher perceived aesthetic in the network-generated artifacts from a more shifting vantage point. This perception of aesthetic was dynamic perspective in two respects. First, instead of teacher focusing on each level separately, Martha's perspective was more changing, particularly shifting from the whole group level down to individual. Secondly, the sense of aesthetic comes from her seeing the group artifact as composed of moving or movable parts. Unlike static view of aesthetic, where the group artifacts is view as a complete picture with still symmetries, this notion of aesthetic comes by perceiving the smaller parts of the aggregate image as converging or moving in a way that creates a pattern. While the pattern itself may be fixed, the composition pieces of the larger mathematical object that are perceived by the teacher as being in motion.

One episode during which Martha and the researcher planned the *Linear Family* lesson illustrates this dynamic aesthetic at work in pedagogy. As worked they together, the extent to which Martha planned for some emotional impact of the image on the students became apparent. After she and the researcher finished creating the aggregate image of a *Linear Family* (Figure 4.1) with a shared root, they had the following exchange:

MARTHA: We will get some definite "cools" from [interrupt]

RESEARCHER: What's cool about that?

MARTHA: Well that all these strange equations come together at these two points to make this visual image...this nice visual image. That having things,

> it's just like a dance where they have everybody off doing things and they come together for a second [increased vocal intensity] and then go back out again. It's the same image, except it is with points on a graph. Here is where they coincide and they go off and do their own thing. They go off and do their own thing again, boom. And so you're seeing all these shapes, and do they have anything to do with each other? Boom! They do. And then, boom! They are on their own again.

RESEARCHER: What about that? We didn't highlight that.

MARTHA: We'll just let them talk about it. They will go "cool." Someone will.

(Tape No. U04011301, 1:28:00)

What made this a dynamic perspective was the fact that the mathematical construct to be taught was conceived as a composition of smaller moving (or moveable) parts. Thus, Martha used the "starburst" hand gesture to describe the family of linear functions. In her words, she could "see all those lines coming in," even the ones that were not physically visible. Furthermore, it was the form of the whole that determined the valid movement of the parts. Parts were seen as valid only if they fit well into the whole image; if not, then the individually generated artifact was seen as moving incorrectly. In this sense, dynamic perspective of aesthetic was used as a way to assess the level and quality of the students' understanding. Incorrect solutions were those that seemed to be moving in the wrong direction or off course, relative to the larger picture. The dynamic perspective emphasizes that the aggregate is not simply determined by validity, but that mathematical validity of individual student-generated constructs is determined by the aesthetic of the whole-group artifact.

Analysis of the data showed that this perspective went beyond the dichotomous perspectives, group and individual, but instead positioned the teacher from the side of both levels, sequentially. In the dynamic perspective the visual image of the aggregate or the teacher preconception of it, is perceived in such a way that it dictates specific student actions. The parts, which were the individual student contributions, were dictated by the appearance of the whole group generated artifact. Which individually created artifacts did not seem to contribute to the larger picture? Did any seem out of place?

The dynamic perspective was also emphasized in a class discussion of the *Linear Family* lesson. In this case each line was described as moving in a particular way as to complete a pattern, thus projecting a dynamic view of the generated artifact.

Commanding the class' attention, Martha asked, "So now, what do you think is going to happen when we all send our equations that go through this point up here [pointing to the overhead]?" "It's going to have lines like...[making multiple slant gestures (of varying angles) with her hand]...going through the [point]," explained one student. Martha clarified, "So you think it's going have them all going like this [imitating the student's hand gesture] through the point? They'll all look like this?" At that point the class discussion became more lively with several students offering predictions, talking over one another.

STUDENT$_1$: They'll all touch the...[interrupted]

STUDENT$_2$: They're all straight lines, so they should all touch the point...[interrupted]

STUDENT$_3$: I think they'll stop at (3,0).

MARTHA: So you think it's going to look like that [making a v-shaped gesture with both hands]? Alright. I'm glad we talked about that, because I want to see what happens.

Upon seeing the image of the linear family that the class had generated (Figure 4.4), one student commented, “All the lines are collapsing.” “So you think it kind of has some representation or gives you the feeling of something collapsing?” Martha asked, at which time several other students chimed in unison, “Yeah!” “Yes,” said another student, “They are all going to one point.” “And these are all going to one point?” Martha asked again. “Well actually they’re not going to the point. They’re going through it,” another student corrected. Martha clarified further, “Yeah. In our picture it looks like they‘re all kind of connected together but really in real life you are saying they just go through it [point].” The artifact was perceived as multiple lines simultaneously moving through the point. (Tape No. U04011602, 27:00)

The teacher’s references to the activity of the student-generated constructs with words like “going” and “collapsing” in these quotes gives the sense that she is perceiving of the artifacts as dynamic constructs. Here again, Martha’s use of language like “go off,” “come together,” and “collapsing” all point to this sense of her perceiving the class-generated artifacts as being in motion—dynamic.

After the brief discussion, the students worked individually, sending to the network graphs of linear equations through the given point. Martha and the researcher walked around the classroom, helping some of the students. When the students were finished, a few of them were eager to view all of the graphs on their individual calculators, after which time one of them exclaimed (perhaps, facetiously so), “Wow!” Another student chuckled as he looked at the graphs on his own calculator. Two other students gave each other a ‘high-five’ gesture. Surprised at their reaction, Martha queried, “Oh. So you looked at them already? Ok. Let’s get them.”

It was as if she wanted to save the aggregate image for a final ‘punch line.’ The two boys, obviously friends, sat at two different tables, not far from each other; yet their high-five gesture implied that they had formed their own small group. Aesthetic, a shared goal between them, was jointly valued and defined in their small ad hoc group. This also underscores the ubiquitous nature of aesthetic perception in the environment in which HubNet supported generative design is enacted.

The emphasis in Martha’s explanation of aesthetic is on “dance”. This dynamic perspective of aesthetic was seen as a choreographed perpetual motion of individual activity. This, in fact, is what she thought of as “cool”, as representative the phenomenological experience of mathematical aesthetic and a useful tool for assessing student understanding.

The aesthetic of the group image acted effectively as a determinant of the mathematical validity of the student work, the activation of which required a dynamic focus on the artifacts—whole group downward to individual. Students’ actions, independent of one another, were brought into coalescence around a central mathematical idea. Furthermore, the mathematical construct itself serves as an ordering mechanism for student activity engagement. The following are some quotes from implementation phase of *Rule for Points* and *Product of Lines* lessons (described in more detail in the next section) that illustrate the work that the teacher’s aesthetic perceptions do in the dynamic perspective. In every case there is a group generated pattern or image (e.g., line of points) into which each of the individual student-generated constructs should fit:

- “Ok. Do you think everybody has gotten there [where x and y are negative]?”

- “What about the white car? Do you think that it’s in a place where x is positive and y is negative?

- “[Directed at one particular student] If you pick points and get points all over the screen, we don’t see a pattern. We don’t see anything interesting mathematically?”

- “[Responding to a student’s comment] So you’re saying that the little cars [points] go up two, one over, up two, one over all the way, except for these [pointing to errant points]?”

- “I’m going to get [highlight with cursor] these [seemingly errant] points and look at them and see if they’re in the right place or not.”

- “So whoever picked this point...and you know who you are...you would probably need to move it on up, because it doesn’t fit the pattern, right?”

- “I have a question. What about this little point over here in the negatives? Does it fit?”

Phrases like “gotten there,” “the right place,” and “fit the pattern,” among others, all suggest a shifting downward in the teacher’s perspective from whole group level to individual, not a fixation on only one level at a time. As the above quotes also suggest, this shifting is done to evaluate an individual artifact relative to the larger group.   Each of these quotes suggests that, like in the case of dynamic aesthetic, it was the appearance and location of the point relative to the whole perceived pattern of the image that determined its validity in a similar way to dynamic notion of aesthetic. The check was as much a question of its coherence of the image as it was a check of mathematical validity. Moreover, the aesthetic comes from the coordination of the actions (or perceived) actions of student-generated artifacts in a particular way, as to create a coherent and cohesive image. Dynamic perspective of the aggregate image was used to verify the work of individuals. If an individually generated artifact did not fit in with the rest, it was seen as incorrect. Mathematical validity of student-generated constructs is determined by its fit in the whole group image.

Emergent perspective and aesthetic

As a result of repeated coding and discursive methods one other type mathematical aesthetic instantiated by network-supported generativity could be identified, emergent aesthetic. Emergent perspective is characterized by the notion of a global mathematical pattern as emerging from subroutines from each student, emergent aesthetic. While the dynamical perspective shifted the teacher's focus from group to individual or higher-to-lower levels, the emergent perspective operated in the exact opposite direction. However, sometimes level of perspective of the projected artifacts oscillated back and forth between individual and whole group. For this reason it was sometimes difficult to make the distinction between dynamic and emergent perspectives, but by definition, emergent was categorized as beginning on the individual level. The difference is rather subtle, but in emergent perspective the teacher's focus customarily shifted from individual upward to whole-group levels of participation, where the whole artifact was seen as emerging from the coordination of the individual parts (student contributions). Again, like that of dynamic perspective, the aesthetic of emergence came from the coordination of the individual parts acting independently.

It is here that there was found an interaction between levels of perspective of aesthetic and the type of generative lesson that was being implemented. The *Linear* and

Quadratic Families artifacts result from Martha giving a goal to the entire class, while the *Rule for Points* and the *Product of Lines* artifacts come about by rules given to individual members of the class. Table 4.2 suggests that the former two lessons invited dynamic perspective, but not emergent. The later pair of lessons required as fluid a perception of generative artifacts as the others, incorporating the dynamic perspective, too. The *Rule for Points* and the *Product of Lines* lessons, however, initiated a perspective of the generative artifacts that flowed in both directions—from aggregate to individual, as well as individual to aggregate.

Since all of the individually generated constructs (student contributions) in the whole group artifact emerge into a pattern, they share aesthetic qualities with other emergent mathematics structures (e.g., fractals), patterns from independent constructs and/or mathematical routines.

There were several occasions in which Martha asked the students to look at the projected image and decide whether all of the points, each one controlled by an individual student, seemed to fit into the picture in a coherent way. Aesthetic is captured in the self-organizing pattern of the aggregate of the student generativity in the class. This is indicative of the subtle distinction between Dynamic and Emergent perspectives. Because emergent level also involved shifting perspective between individual and group, it was difficult for the researcher to make clear distinctions. Below are several quotes from Martha on separate occasions during the lessons, which emphasize this emergent view of the artifact. On every occasion she referred the students to the overhead screen.

All the students in the class had just re-entered the activity as points on their individual calculators and as icons on the overhead screen. The students had just completed a sped-up version of the *Rule for Points* activity. Since this was the algebra II class, they all well aware of the construct of linear equations. It did not take long for them to form the two different lines on the overhead screen with their icons according to the two rules that Martha had given them (*y*-value is one more that the *x*-value, *y*-value is two less than the *x*-value). Martha had several quotes from the following class activity that displayed her perception of emergence in the mathematical constructs.

After a few minutes of the students moving their points around on the up front display screen in the *Rule for Points* lesson, Martha asked, "What's happening? What's happening with this shape [pointing to the screen]?" One student responded, "What shape?" "Do you think we're not getting a shape?" asked Martha. "I don't think we're getting a shape or anything interesting," answered the student. At about the same time another student called out, "A diagonal!" "Oh. I see it," said the original student. Martha tried to continue the discussion, but some students were still moving their icons around the screen, even after they had found their correct location. Martha complained, "Guys, get where you need to be and stay there. I don't care if [your icon] is on top of [another one]. Just get there and stay there. I know you're having to battle for a space, but..."

Martha continued the discussion, "What do you think? Does it look like that might be a line, too?" She was referring back to the previous activity in which she asked the students to move their icons to a place where the *x* and *y*-values where the same. Students could clearly see a line then. She wanted them to determine whether the geometric pattern that they were attending to also made a line.

All of the icons were in quadrant one, except for the one that she had asked for from a student earlier. It was in the third quadrant. She pointed it out and asked, “What about this point here? Do you think it’s on our line?” “No, I don’t,” said one student. Martha then took out her yardstick and lined it up and said, “Yeah. It looks like it’s on the line.” The students agreed.

Later on in the lesson, the objective was for each student to take a given point on the Cartesian plane and reposition it according to a mathematical rule. Martha would sometimes gives hints to all or individual students to help them meet the requirements of the rule.

“Everyone look at your paper. I gave everybody an x-value. Look at your paper. You don’t get to go anywhere but there for a little while, but that x. If you look at your screen it will tell you what x [value] you’re on.” She continued to explain some of the functionality of the system, how to move around, change the “step size” of each movement of the points, and generally, how to interact with the network system. “Ok now, as it says on the paper here, I want you to make your cursor go to the point where your y is one more than your x. Your x is forced so then you have to decide where your y is one more than that.” Quietly, the whole class worked on their calculators. On the screen up front, the points shifted around randomly with Martha occasionally reminding the class of the rule, “Your y has to be one more than your x.” About three to four minutes later the following image (see Figure 4.5) appeared on the screen up front. (Tape No. U03111401)

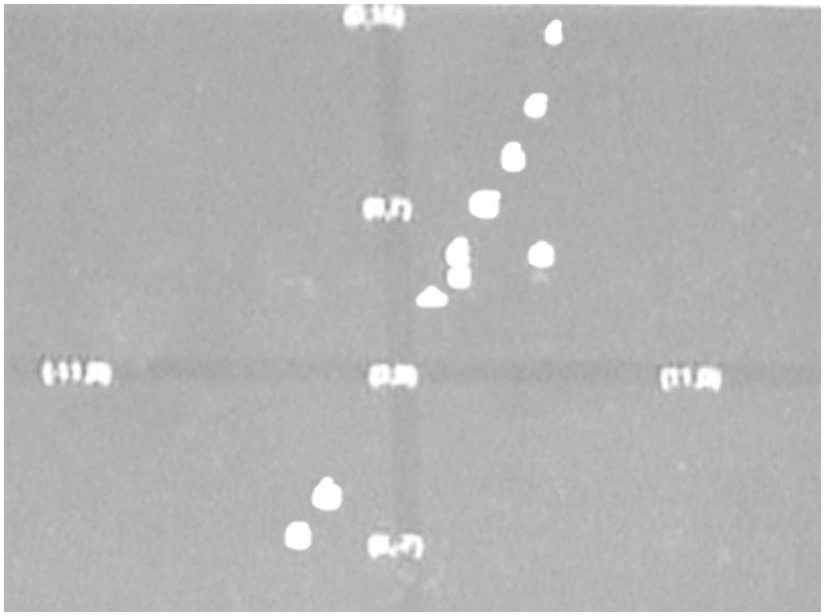


Figure 4.5 Still image from a video clip of the *NetLogo* screen appearing on the up-front display during the activity, “move to a place where your *y*-value one more than your *x*-value.”

Martha’s Algebra II class completed the *Rule for Points* activity and then moved on to the *Product of Lines* lesson. She wanted the students to use the same *x*-value and move their icon to the location that was the product of the two previous *y*-values. As the students moved their points around, Martha called out, “Ok. Now we’re starting to see something. Has everyone else moved? Oh, we’re still getting people on [appearing on the overhead screen]. Are some people still moving their cursors [points] or are we all there? It’s looking pretty good up there. What kind of shape are we getting up there?” One student mumble, unsure of himself, “ A line?” Martha responded, “You think that looks like a line right there?” Several students then tried to answer at once. “No. It’s a um...,” said a student upfront, attempting to recall the name of the shape seemingly familiar to her. Another student, also familiar with the shape, did not even attempt to answer Martha’s question. He just wanted to ensure that the shape was correct, and said, “Someone needs to move his brown car [move the point into the right spot].” “Yeah. Whose brown car [is that]?” shouted someone else. Completing the second student’s response, another student chimed in, “It’s a parabola.” “Yeah,” agreed several others simultaneously. “So Heidi’s saying that she thinks it’s starting to look like a parabola,” Martha repeated. “I think it’s going to be a parabola,” the student clarified. Several students continued to move their icons around on the screen, seeking to complete Martha’s challenge. “Yeah. We still have a square that’s moving and a car that’s moving,” Martha said.

Once the two lines of points were formed and projected onto the upfront screen, Martha directed, “ So we have one line that equals x+1 and the other line that equals x-2. Ok, I want to multiply these two lines together.” She then asked the students what they thought it would mean to multiply two lines together. One student explained, “It’s like the distributing thing we’ve been doing, like he [referring to another students comment] said with last, outer...” Martha reiterated, “ Ok, you kind of do a ‘foil’ thing, ok.” She walked over to the white board and began to multiply the expression $(x+1)(x-2)$, resulting in the polynomial, x^2-x-1. Then pointing back to the two linear factors she asked, “This is the factorization of...? Well we factored something into linear equations [from previous classes]. What was it?” After leading the students for a moment, she finally got some of them to answer, “Quadratics.” “Oh, yeah,” a couple of students in the front said. She continued, “Ok. I want to see what it means to multiply your points.” “So we’re just putting both of these things together?” asked one student, seeking clarification. “Yeah. We’re going to put them together, but your x has to stay the same remember,” she responded. Then the students were given a third and final point and asked to move it to the coordinate where the *x*-value was the same as the other two, but the *y*-value was the product of the other two *y*-values. However, at this time it became evident that some of the students' creativity in this activity pushed the limits of the network capability. Because some of the students chose fractional *y*-values (e.g., one-half), the products were smaller fractional values that could not be displayed on the network. Martha, with a quick impromptu reaction, suggested, "Oh. Then round up to the nearest half." She looked back at me in the back of the classroom and smiled. We had forgotten to consider this one limitation of the network. After several minutes, some prodding by Martha, and noisy interaction between the students, the screen up front began to rearrange (itself). Below (Figure 4.6) is a still image from the video clip showing the pedagogical artifact of Martha's class in which students multiplied *y*-values of points satisfying the two rules: *y* is one more than *x;* and *y* is two less than *x*. (Tape No. U03111401, 07:00)

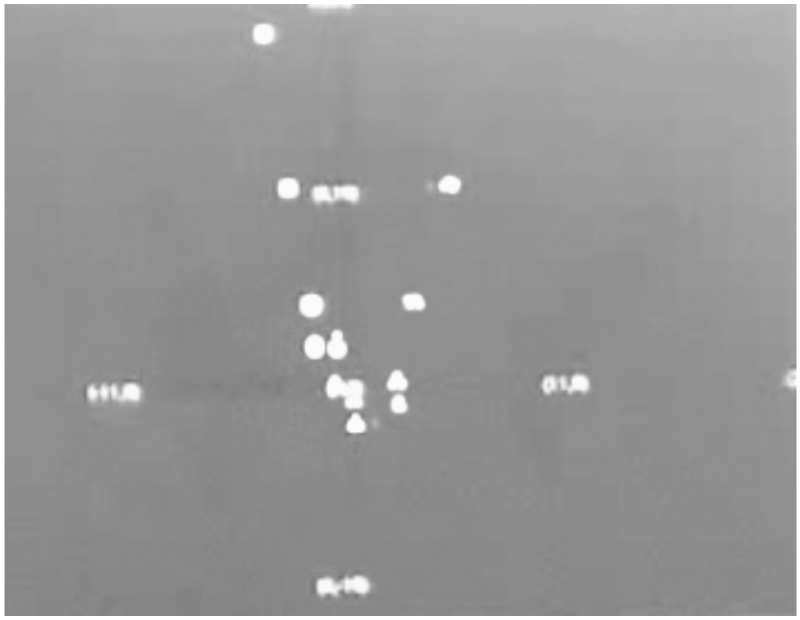


Figure 4.6 Still image from a video clip of the *NetLogo* screen appearing on the up-front display during the activity, “move to a place where your *y*-value is the product of your previous two *y*-values.”

Martha’s intention was to give the students a personal experience with the mathematics constructs they were studying. It was the personal experience of a participatory simulation that helped students to connect the local activity of individual agents to the global pattern being produced. By asking, “What’s happening with the shape on the screen,” Martha seemed to be pushing the students to look back and forth from their own individual work of finding their location on their calculators to the line that was taking shape on the up front screen. This is a first-hand experience of emergence. Because Martha perceived the aggregate image as an emerging pattern, her plan was for the students to see that as they moved around the image would begin to take shape, and a discernable pattern would come into view. The following is an episode from the implementation of the *Product of Lines* lesson that gives an account of how the activity played out in the classroom.

In having the students predict the aggregate image to be projected on the upfront screen, Martha forced the students to oscillate back and forth between their own personal view space on the calculator, where they engage in the simulation activity, and the screen in front of the class, where they engage socially. She seemed concerned that the students shift perspectives, high to low and vice versa, so that they would understand the effects their individual work had on the rest of the class and how the whole group actions influenced their thinking about their own personal space.

Table 4.2 is presented to give the reader a sense as to the relative weight of qualitative evidence in support of the notion of static aesthetic perceived in the emergent generative artifacts, particularly from the individual perspective. Number of Static records (Individual and Group) is over twice that of shifting perspectives (Dynamic and Emergent). This is possibly the result of the fact that it was easier to identify aesthetic in the simple terms of a static image. Also, it may have been easier for Martha to initiate dialogue with her class about interesting patterns of the whole group image than to delve too deeply into the exact nature of the aesthetic. Most of all, the sheer number of individually constructed artifacts within the group image ensures that this category would be higher than all others.

Table 4.2
Number of aesthetic-related records found in each level of perspective.

Level	Description	# of records
Individual	Perceived aesthetic of individual student-generated artifacts.	**68**
Group	Perceived aesthetic in whole class-constructed artifacts.	**42**
Dynamic	Perceived aesthetic of whole class-constructed artifacts as a composition of individual parts in perpetual motion.	**25**
Emergent	Perceived aesthetic of whole class-constructed artifacts as emerging from individual actions.	**18**

In a related way, the difference between the activities was found to be a co-related facet influencing the way the teacher used aesthetic perception to help shape instruction. Table 4.3 shows that the Emergent category is relevant only for the *Rule for Points* and *Product of Lines* lessons, because these two were the only ones that were about emergence in the classical sense. All other levels of perspective contained at least one record for each of the lessons. Again, the individual perspective total was high because of the nature of these activities, which all relied on individual student-generated artifacts.

Table 4.3
Number of aesthetic-related records found in each level of perspective by lesson.

	Individual	**Group**	**Dynamic**	**Emergent**
Rules for Points	22	15	13	14
Linear Family	31	15	5	0
Quadratics Family	25	10	7	0
Product of Lines	11	5	1	3

From Martha's vantage point the students were demonstrating their own particular thinking as depicted by the individually generated artifacts, the coordination of which created one single mathematical construct, a line in this case of the *Rule for Points* lesson and a quadratic in the case of the *Product of Lines* lesson. This is the essence of an emergent generative artifact.

CHAPTER FIVE

IMPLICATIONS AND CONCLUDING REMARKS

The observations presented in this case study are intended to elucidate the inner working of a generative instructional technology that seems to make visible the aesthetic perceptions of mathematics in both the teacher and students. The specific questions to which this study responds are as follows:

- To what extent does the teacher invoke mathematical aesthetic perceptions in (1) designing and (2) implementing HubNet-supported generative lessons?
- How does the teacher mediate between various levels of activity engagement with the technology to invoke aesthetic perceptions?

One secondary mathematics teacher was observed and interviewed as she prepared and implemented network-supported lessons for her Pre-algebra and Algebra II classes. The researcher conducted a case study within the context of a larger design experiment in order to provide a qualitative dimension to the mixed methods research design. The case study was to attend to the teacher's role in network-supported generativity, while other parts of the design experiment focused mostly on student learning (ISME, 2002).

Because of the projection capability of the HubNet technology allowed class-generated artifacts to be projected to the entire class and became a mediating device for the mathematical discourse. Data collected in the case showed the teacher using aesthetic perceptions of the mathematics represented by the artifacts as guides for what to talk about in the class. The teacher also made plans around the appearance of the artifact to discuss certain features of the artifact that she found "interesting." She also anticipated the ways that student would react viscerally to the appearance of the artifact. For example, she sometimes would predict, “We’ll get some ‘cools’ from this.” Phrases like this appeared often in her conceptualizing and planning sessions. Also, in conceptualizing how the HubNet activities would match her given curricular objectives, she would often set affective goals for the lessons. Some of the times the teacher explained her goal for the lesson, at times, articulating mathematical aesthetic explicitly, explaining that she wanted the students to be surprised or somewhat amazed by the visual manifestation of the mathematical constructs through the class-generated artifacts.

In a related way, the teacher compelled students to focus on the projected artifact on both individual and group levels of perspective, separately, in order to perceive aesthetic in the mathematical object. This static perspective engages an equally static aesthetic of the artifact. The projection of the group-generated object was seen by the teacher as beautiful, seemingly for its geometric symmetry. “The fine geometric beginnings of a mandala” was her description for one of the artifacts. Use of phrases like “nice patterns” to describe other group-constructed artifacts impels the implication that it was her perception of the artifact as a nice visual, but static image that invokes her sense of aesthetic appreciation. Additionally, she pointed to the aesthetic to be perceived by oscillating perspective between individual and group level, first, from group to individual, in order to perceive a dynamic aesthetic in the artifacts, and second, from individual to group, in order to perceive an aesthetic of emergence in the artifacts. The case revealed that aesthetic perceptions of mathematics were operative in the practical design constraints of generativity through the teacher’s coordination of various vantage points from which to view the emergent generative artifacts—individual, whole-group, dynamic, and emergent—as well as in her ways of referring to them. It was the design of the HubNet technology that created opportunity for aesthetic to be revealed, thereby, framing the teacher's use of the network. The class discussions were centered on the appearance of the upfront screen and the students' own calculator screens, which displayed the projected image of the class' activity, shifting perspective to view natwork-generated artifacts as dynamic and emergent phenomena.

It was found that planning lessons around participatory simulations on the network acted as a generative "template" by which to shape instruction. As a consequence, the aesthetic perceptions of the teacher emerged as a factor in the design and implementation of the network-supported generative activities. The interpretation of these findings imply that aesthetic perception was not only an influencing factor, but can also be thought of as a fundamental structuring tool of generative activity design. It is thought that as the teacher considered the appearance of the artifacts produced by the network, the technology and the generative activity designs for it made aesthetic visible and a significant part of the mathematical discourse in and out of the classroom.

One limitation of the research design reported in this thesis is that it discusses one single case that might represent an atypical scenario. In addition the elaborate theoretical framework cited early on in the course of this investigation, constituted an evolving interpretive framework, which, some may argue, made the researcher's finding of aesthetic perceptions in the data more likely. However, these facts are consistent with design experiment methodology (Brown, 1992; Collin, 1992). In fact Brown (1992), an early pioneer of design experimentation in educational settings, argues that design experiments have an implied tone of advocacy for the design, in that the methodology involves suggested revisions of it in order to achieve the desired educational outcomes; otherwise, there would be little impetus for research the design at all.

Another limitation is that the interpretations of findings are highly researcher dependent. It was the researcher's perception of mathematical aesthetic framed the context of the case, what would be talked about, what would be looked for in the data, and how it would be analyzed, all of which were colored by the theoretical lens of the investigator. Although this is consistent with other case studies (Moss 1994, Stake, 1995), the design experiment methodology, nevertheless, made for a more-extensive-than-usual collaboration between teacher and researcher, to such an extent that some might contend that the data are 'contaminated', suggesting that the finding say as much about the researcher as the case itself. This might have been averted some by a revision of the research design that includes the establishing of a more rigid coding scheme prior to collection or review of videotaped data.

Mathematical Aesthetic in Instructional Practice

In attempting to understand how the notion of mathematical aesthetic might get instantiated within the context of education, Sinclair and Watson (2001) question whether aesthetic experiences in mathematics "can be made available to students (p. 39)." They propose general descriptions of what aesthetic might look like in instructional settings, suggesting that teachers structure student engagement with mathematical activity in a way that "results which might otherwise seem commonplace emerge as surprising special cases (p. 40)." Wang (2001) further suggests, "The classroom needs to be arranged in such a way that transformative and transcendent qualities of the aesthetic can be incorporated to embrace the unexpected, and daily experience can be crafted into something extraordinary—an experience" (p. 92). Summarily, these researchers characterize an aesthetic experience for students as having such distinguishing features as surprise, induce affective responses, evoke feelings of pleasure, and dramatic.

In some of the vignettes detailed in Chapter Four, aesthetic critique of network-generated mathematical objects was found to be reflected both in the teacher's own understanding of the mathematical idea and in her design of an instructional activity on the network to project an "interesting" visual image of the idea. The projection of the whole group artifact mediated the mathematical discussion in the class and allowed aesthetic to emerge as a part of that conversation. In the case study, most of the teacher's were focused squarely on the upfront space. It was her perception of mathematical aesthetic that was found to influence the activity engagement with the network.

Analysis of the data from the case study also revealed that aesthetic operated, not only within her own perceptions, but also overtly in the pedagogical practices of the teacher. In planning HubNet lessons, she made plans to state overtly her affective goal in the lessons, for students to be "wow" by the mathematics they were creating and what they would see on the upfront display. She wanted her students to experience mathematical aesthetic, and the projection capability of HubNet was what she used to achieve that goal. What is new to consider are the ways in which the teacher's aesthetic perception of mathematics does real work in instruction in general and more particularly, in choosing topics of the class conversation and in planning for affective results for student engagement. Would aesthetic consideration play as big a role in other instructional setting? If so, then what is the nature of that aesthetic? What ways would it appear differently than in network settings?

The work done by the teacher's aesthetic perceptions in the context of mathematics instruction resonates with the features of mathematical aesthetic, as detailed by Sinclair (2002). As she explains, the evaluative feature of mathematical aesthetic "concerns to aesthetic nature of mathematical products such as proofs and, more specifically, the judgments made about which products are most beautiful, most elegant and most significant (p. 13)." It is this evaluative nature of aesthetic that is prevalent in Nelson's (2001, 2002) "proofs without words" (p. v), where visual images that count as proofs open the door for elegance to be a part of the conversation about the aesthetic of competing explanations. In addition, Sinclair also describes the motivational feature of mathematical aesthetic as providing "the initial attraction to a situation or problem that provides the motivation for pursuing a solution to that problem (pp. 47-48)". She explains further that aesthetic has been described as compelling "not only the unconscious choice that leads to mathematical discovery, but also the more general choices about which investigations to pursue (p. 48)."

The case study illustrated a number of ways in which Martha made choices between individually created artifacts. She called for the "most interesting" student construct during the implementation of the HubNet lessons. This amounted to specific challenges in some activities for students to find the most unique or creative solution. Some constructs, however, the teacher judged as trivial and of lower value. She used phrase like “this is and interesting one” or “I seeing some interesting things up there” as a way to single out specific solutions that she found aesthetically pleasing in order to engage the class discussion. She described other solutions as “boring ones.” In these ways, the teacher exhibited examples of evaluative and motivational features of mathematical aesthetic. The aesthetic motivation was then passed along to the students as additional constraint in the activity. The case gave a sense as the extent to which aesthetic perceptions of the teacher dictated hers and the students’ behaviors in the network space. Surprisingly, however, data from the case study also illustrated how it could be limiting to student activity engagement. Sometimes the teacher compelled students to send her chosen parameters for the solution in order to remain true to her own vision of the final group artifact. So aesthetic perceptions of the teacher did not always have a positive effect on the class activity.

In a related way, Davis and Hersh (1981) recount a recurring theme of the aesthetic of mathematics, “ordered patterns from chaos (p. 172).” This is also illustrated well in Herstein’s (1964) notion of the aesthetic of group theory, where he notes the impressive (mathematically speaking) feature of algebraic structures that they can reduce a large set of qualitatively different forms to one single phenomenon by finding an analogy that links them all. He suggests that much of this important branch of modern mathematics is concerned with finding and explaining how different objects, which have obvious qualitative differences, can be lumped into one equivalence class with a few, very general commonalities among them. They are all different in some respects. On another level, however, a group theoretical level, they are simply variations of the same form. Once the layer of qualitative adornment is lifted, the objects are indistinguishable.

Herstein explains:

> The systems chosen for study are chosen because particular cases of these structures have appeared time and time again, because someone finally noted that these special cases were indeed special instances of a general phenomenon, because one notices analogies between two highly disparate mathematical objects and so is led to a search for the root of theses analogies. To cite an example, case after case after case of the special object, which we know today as groups, was studied toward the end of the eighteenth, and at the beginning of the nineteenth century that the notion of an abstract group was introduced. The only algebraic structures, so far encountered, that have stood the test of time and have survived to become of importance, have been those based on a broad and tall pillar of special cases. Amongst mathematicians neither the beauty nor the significance of the first example, which we have chosen to discuss--groups--is disputed. (p. 27)

In other words, this one topic in mathematics, Group Theory, has its aesthetic in finding ways to construct a single form from a seemingly diverse set of objects by finding their general commonalities in what Herstein termed "the root of analogies."

While it is well known amongst mathematicians that there exists an entire family of infinitely many lines with the same *x*-intercept or quadratics with the same two roots, it proved to be quite a novel concept to students. Moreover, when this mathematics construct appeared as an artifact of student activity in the classroom, not only was the obviousness of its veracity revealed, but also its surprising elegance in the ordering of what seemed to be disparate moving objects.

It is in this context that aesthetic in mathematics education bears some resemblance to the order-chaos notion of mathematical aesthetic in mathematics—the ordering of a seemingly chaotic and disparate set of constructs into a single pattern. It was in the group artifact, moreover, that the teacher was most particularly clear in articulating her sense of aesthetic appreciation. In the *Linear Family* artifact, for example, she could see all the lines coming from different angles to converge on a point and then diverge. Her perception of aesthetic seems to come close to the order-chaos definition. The use of copious examples from student-generated work helped identify mathematical patterns and relationships on a broader scale.

The generative teaching technology was important in that it created a space for the mathematical ideas to be expressed creatively, as dictated by the student generativity, subsequently allowing the teacher to emphasize more aesthetically salient features of particular mathematics constructs in the student created artifacts. As the HubNet allows for its appearance and salience of aesthetic as a topic for the mathematical discourse in the classroom, it affords a unique perspective of mathematics pedagogy as an experience of artistic critique. What is unclear, however, is the extent to which these findings can be extend to other instructional innovations or settings that produce pedagogical artifacts. The essential questions are: In what sense can aesthetic be seen as a significant part of the instructional design and implementation process, in general? What is the role of aesthetic in other instructional technologies (non-silicon-based technology included) that require the teacher's engagement with student-generated constructs? To what extent are mathematical aesthetic perceptions integral to teachers' decision-making, behaviors and utterances in practice?

To ask why a teacher chooses certain student-generated mathematical solutions, proofs, conjectures, or objects to point to in class discussions, it is conjectured, is to ask about the aesthetic perceptions of the teacher. The implications for reform are that making overt the aesthetic undertones in mathematics pedagogy may provide more viable learning and teaching trajectories for the traditionally underrepresented students in mathematics and leverage those modes of discourse that are customarily under-valued in the educational arena.

I argue that aesthetic is relevant to the discussion of mathematics education because of the reform goal to havse more (all) students of school mathematics find aesthetic appreciation in mathematics (NCTM, 2000). It is, therefore, important to investigate questions relative to mathematical aesthetic and to determine ways that it might be emphasized through the affective goals of curricula, while maintaining the integrity and intellectual rigor of classroom mathematical practices. When someone says "cool" or "that's beautiful" in reference to a mathematical idea, to what exactly is he or she referring? What is the role of mathematical aesthetic in mathematics teaching and learning? The findings in this investigation suggest that arenas where the research community might continue the search for answers to these questions include teachers' aesthetic perceptions of mathematics, classroom mathematical discourse, instructional tools, and mathematics curricula.

Aesthetic extension of Generative Design Frameworks

Several times during the course of this investigation aesthetic was allowed to mediate student experience in mathematical exploration, consistent with Sinclair's (2001) characteristics of mathematics aesthetic. On several occasions during planning, Martha said that she was prepared to encourage the students to "try [different] things" or try to send the most "interesting" student-generated artifact, in responding to the network activities. This suggests that she not only had a high comfort level, but a deeply ingrained attitude about mathematics and mathematics pedagogy as a "playful" and aesthetic-driven experience. Martha took the position that "it [mathematics] makes a lot more sense [to students]" when they are allowed to explore. These are indicative of a deeply held belief about both mathematics pedagogy, as well as mathematics practice.

Yet, in order for her to take this position, she had to be willing to acknowledge the multiple perspectives of any given mathematical task. Such a view is at the core of the notion of "space-creating play" in generative design (Stroup et al, p. 191), where mathematics instruction is designed to broaden the space for student exploration, providing for many viable solutions to one problem. The space-creating mechanism serves to illuminate the space of viable responses and to provide boundaries to limit incorrect ones. This is what is meant by "space." It is both mathematical and pedagogical, created by the actual participatory simulation itself and any extra constraints imposed by teacher's challenge.

By asking for the students to send to the network their most "interesting ones [student-generated construct]", Martha proffered aesthetic as an explicitly stated "object of the game." Such a request calls for students to make a value judgment about the quality of their artifacts. In this manner, pursuit of aesthetic is an "organizing form of activity" (p. 191). A competition of individual inventiveness can begin to dominate the classroom atmosphere. The goal of the competition is to find the most creative and novel solution, projected on the screen up front as an artifact of ones mastery.

Sometimes, however, students would limit themselves to a much smaller set of possibilities, and Martha would have to encourage the students to explore more. To enact the space-creating play, Martha helped expand the space of possible student engagement, although it was not actually the space that expanded, but the students' conception of it. She used the conception and planning phases to anticipate whether she would need to provide extra guidance and hints to encourage the students to explore more than just their initial responses.

By predicting how students would interact with the activity during the conception and planning phase, Martha could pre-examine the "contour" of the space, thus, enabling her to consider the range of possible student responses. Mathematics teaching and doing were indistinguishable in this context. In a similar manner as expressed by Ball and Bass (2000; 1999) and Ma (1999), the teacher's tasks, relative to preparing for HubNet generativity, were as much mathematical as they were pedagogical. She also predicted whether students would interact with activity in such a way as to display the breadth of possibilities. She wanted to "fill" the space. The generative artifact was used as a type of assessment of students' ability to access less common, more creative solutions.

The episodes in Chapter Four illustrate that the HubNet makes aesthetic visible in ways that connect it directly to and gives further empirical corroboration for the generative activity design framework proposed by Stroup et al (2005; 2007). The evidence also suggests a need to augment the existing framework for generative activity design to include more developed notion of aesthetic, extending their description of creativity and inventiveness of "play" (2005, p. 191). The episodes reported herein, organized in terms of kinds of aesthetic perception (static, dynamic, and emergent) and arranged by phase of pedagogical development, suggest that aesthetic perceptions of the teacher presented distinctive and significant avenues for aesthetic engagement as a part of discourse in instructional settings. The existing framework for network supported generative design might productively be extended to emphasize the work that aesthetic perception does in redefining "space-creating play" and re-shaping modes of legitimate participation. Moreover, I argue that this can begin to establish for the research community ways of talking about mathematical aesthetic, not simply as an ethereal or "removed-from-practice" notion, but as an active construct that does work in teaching in ways similar to those documented in this thesis.

What is new here is the characterization of the manner in which Martha's aesthetic perception of mathematics could be seen to re-shape student engagement with the activities by emphasizing what she thought was "interesting", i.e., aesthetically pleasing. In this sense, space-creating play was tantamount to the search for the most mathematically "interesting" solution (construct). This is also related to Sinclair's (2001) notions of motivational and generative characteristics of mathematical aesthetic in that aesthetic is thought to direct both the initial choice of mathematical work and the actual process of mathematical inquiry.

The teacher's aesthetic perception of mathematics was important to consider here, because it seemed to constrain her pedagogical decisions in using the technology.

However, once aesthetic was introduced as a part of the activity other perceptions of aesthetic...the dialectical analysis between content and pedagogy ...From this perspective the teacher used her sense of mathematical aesthetic "not just as 'content' to be learned but also as an interpretive framework for learning analysis and activity design" (Stroup et al, 2002, p.198). It was mathematical aesthetic, in fact, that was seen to structure the social sphere of learning.

Connoisseurship and Critique in Mathematics Education

In Eisner's (2002) descriptions of epistemic seeing, he suggested that teachers, who perceive ideas through multiple lenses, are artistically-minded about their domain and have a specialized perception of it. In this case the multiple lenses corresponds to the different levels of perspective from which the student-generated artifacts were perceived by the teacher. Because the teacher's (and students') personal experience of generativity and emergence seemed to take the form of emotionally appealing responses, the ways in which she conversed about mathematics implies a unique perspective of the nature of mathematics pedagogy, which is driven by a metaphor that equates mathematical knowledge creation and validation with the process of artistic construction. This is a sin qua non of Eisner's (1985) notion of educational critique and connoisseurship, based upon Dewey's (1934a; b) assertions that learning is related to the having of aesthetic experiences. This notion lends a philosophical, if not theoretical basis for reflecting on the design and long-term implications of this case study. He claims:

> Artists inquire in a qualitative mode both in the formulation of ends and in the use of means to achieve such ends. The result of their work is a qualitative whole – a

> symphony, poem, painting, ballet—that has the capacity to evoke in the intelligent percipient a kind of experience that leads us to call the work art. My claim is that the paradigmatic use of qualitative inquiry is found in the arts. Another form of qualitative inquiry is found in the work of those who inquire into the work of artists, namely the art critic. (p. 217)

The implication is that both the educator and the education evaluator are viewed as a connoisseur of mathematics, in much the same way as one may be a connoisseur of fine art.

Accordingly, it was found that the teacher, in attempting to convey the phenomenological experience of mathematical, frequently engaged in dialogue reflective of the rhetorical tone of an art appreciation class. Explicitly acknowledging the affective aspirations of the generative activities, the teacher in the case study wanted the students to be able to see an “interesting design”, not for the design’s sake but the mathematical concept that the design embodied. They were like proofs-by-picture (Nelsen, 1993). The aesthetic appeal of the picture was connected to the elegant simplicity of its explanatory power. She directed students’ attention to both the process of the creation of knowledge in the domain, interpreting experiences in ways similar to that of a music critic. The teacher acted as an innovative mathematician and mathematics connoisseur, acting deliberately on the elements of the instructional setting, including the students and technology, to engage the aesthetic and emphasized it in a way that makes it a salient part of the enacted curriculum and instruction.

Conclusion

This study is situated within a literature that seeks to define the mathematical thinking required in teaching (Ball & Bass, 2000; Ball, 1999; Ma, 1999; others). The discussion of this body of work is framed within a larger conversion about finding a context for mathematics practice that is inherent to the field of mathematics teaching, a specialized mathematical knowledge of teachers like that of other fields, such as physics and engineering. The data in this case study points toward a promising direction in the search for a conceptual framework of the educational contexts of mathematics practice, called for by Bass and Ball (2000). As the teacher is compelled to make pedagogical use and mathematical sense of student contributions her role in classroom mathematical discourse is tantamount to that of a mathematics connoisseur and educational critic, fostering student mathematics learning and dialogue about what there is to appreciate various mathematical ideas.

Looking at the practice of mathematics teaching from the perspective described in this thesis, its was found that the social interactions of the classroom were framed by mathematics constructs in some of the ways envisioned in the MS3 dialectic (Stroup et al, 2007; 2005; 2002). In this case, it was the notion of mathematical aesthetic that was thought to structure the social space of the class, as well as the pedagogical practices outside of class. The logic of the argument is that equating mathematics with art pushes up aesthetic perception, a device of mathematics practice, as tool in mathematics pedagogy. Yet, this case study leaves several unanswered questions and unaddressed issues that suggest several future lines of research on the pedagogical function of aesthetic perceptions. For example, while the content topic presented in this study was algebra, one might question the extent to which the HubNet could be used to make aesthetic visible in other mathematics curriculum content areas such as geometry, proof, or number theory, progressing toward a comprehensive theory of the effect of the various aesthetic perceptions on student learning. Will emphasis on mathematical aesthetic in mathematics instruction provide a more open and inviting discourse to students who are traditionally left out of mathematics dialogue? Can the technology be used to engender aesthetic perceptions and affective responses in other scientific domains? Furthermore, the educational research community may find it fruitful to create network-supported generative activity design models that leverage aesthetic perceptions in non-science domain areas as well, e.g., language arts and writing? The implications of a shift towards aesthetic-focused instruction are not known, but studies, like this one, that are grounded in the empirical data from real live teaching practice, will help to uncover them.

APPENDIX A

Tape u03112001 (Interview)

Time	Concept	**Quote**	**Level**	**Description**
0:00	Choosing wording of class acts.	I think it's the words...I don't think it's the understanding...	Ag	
4:56	1.Aesthetic appeal of math and math ped. 2. teach towards generalizations	It was really cool. I didn't think they would come near a generality...	Agg	
5:36	Aesthetic appeal of proofs by picture	They were creating the rule based on the geometric representation which is far better... They could see from the ...	Agg	
7:24	Choosing activity for hubnet	I would like to use these patterns for them to start playing with this (points to hubnet)...	Agg	
15:15	Creating or framing a creative space (rule making, setting activity boundaries)	I want them to get familiar with the space...	Ag	
18:25	A need and focus to teach towards generalizations	They need a little more practice on tricks of the trade to get to generalizations...	Agg?	
19:52	The appeal of the "interesting"	Just to see that tables do create interesting things on graphs...	Agg	

30:00	Considering student capabilities	I think (tentatively) some of them might try this	Ag	
30:28	Constraining the possible student activity	A: How could we keep this from happening? How could we force them to...?	Ag	
30:45	Predicting student responses	Some of them might try it...	Ag	
31:10	Seeking ways to expand the space of possibilities of students' responses (encouraging creative play)	A:Why don't we say "can you find another equation"...	Ag	
31:43	Predicting student responses	By then they will all have gotten somethings that they know how to do...	Ag	
32:40	Aesthetic appeal of special student examples	M: Oh, wouldn't that be lovely...	Ag	
32:54	Alluding to a polished finished product as pedagogical artifact	And all of the lines will have disappeared by then, right?	Agg	
33:17		M: Could I give hints? D: Yes.		
34:04	Predicting student responses	What I think they would want to do at that point...	Ag	
35:09	Encouraging student play	See if you can make another one	Ag	

35:40	Predicting student responses	I think there will be a hand-full that can	Ag	
36:42	Encouraging student play through hints	A: What kind of hints would you give?	Ag	
37:16	Predicting potential caveats for students	It might be confusing for them	Ag	
37:39	Predicting student response	I wonder if they're going to notice	Ag?	
38:39	Encouraging student play	I'm willing to let them just play with it	Ag	
38:42	Defense of pedagogy	Because when you explore and try things...		
40:53	Aesthetic appeal of unique student examples	It would be interesting to see...	Ag	
41:04	Encouraging diversity	We need those kids to get some challenge...we have to have a mix.	Ag	Teaching towards aesthetic uses diversity as a relative strength as opposed to a weakness as it is in the current paradigm of mathematics education.
41:10	Defense of pedagogy	That's the whole point of having them in class together	Agg	
43:48	Pushing for increased diversity	I wonder how I can create a space for that discussion	Ag	
45:50	Recollecting the students' work into a math concept	They could say, "well there is an infinite number of...	Agg	
47:38	Predicting students' emotional response	I bet he really liked having this new kind of approach	Ag	
48:05	Pedagogical critique	Math to him has been ...	Ag	
50:20	Predicting student responses	I think there'll be a hand-full	Ag	

Tape u04011603 (classroom observation)

Time	Concept	**Quote**	**Level**	**Description**
0:00				Function activity
26:20	**T gives s challenge**	[Move] to where both of them are negative	Ag	
27:59	**T unifies s activity around single math concept**	So it looks like we all went down to this quadrant, right?	Agg	
28:57	**T gives s challenge**	You can go any where you want as long as your x equals your y.	Ag	
32:58	**T encourages diversity of s activity**	If you're some place where someone already is, could you move up or down just so it would be more interesting.	Ag	
33:30	**T questions about the upfront space**	T: do you think everybody's gotten to the right place, if you look up there? T: what do you see? Do you see any kind of a pattern from our dots?	Agg?	
35:24	**T questions about the upfront space**	What do you see happening? What do most of those points look like? what are they all doing?	Agg	

38:36	**T discusses incorrect responses**	T: does this point look like it's on the diagonal line that we were talking about? S: no. T: ok. Well, let's check it.	Ag	Responses are discussed with respect to specific rule individually and also with respect to the over all picture (upfront space) of aggregate class activity.
39:29	**Aesthetic appeal of unique (interesting) s response**	Ooh! 3.5, 3.5. Who used a half-step?	Ag	In this type of class diversity of s ideas is praised.
48:24	**T encouraging s to look for patterns in s activity on the upfront space**	We don't get to see the interesting things that happen when we actually look at patterns graphically.	Agg?	In explaining to s how to use the system most efficiently.
48:50	**T questions s about prediction of other patterns in future s activity**	Do you think any rule for an in/out table is going to give up a line?	Agg	Seeing the picture makes all the difference, because it is generated by the aggregation of independent s activity. (epistemic seeing in math ed.)
49:24	**T summarizes s activity in a picture in the upfront space**	We've done two kind of.	Agg	

Tape u03111401 (class observation)

Time	Concept	**Quote**	**Level**	**Description**
0:00	**Hubnet Function activity883333**	This is an activity that will hopefully help you understand quadratics		Hubnet activity for IMP2 (Quadratics as products of linear factors)
11:32	**Teacher gives rule for x-y coordinate**	where your y is one more than your x	Ag	
13:41	**Teacher questions about appearance of upfront space**	T: What's happening on the screen up there? S: they're lining up.	Agg	
22:23	**Teacher gives challenge to students**	Now what i want you to do is, I want you to think about how you can draw, how you can write a function.	Ag	
25:01	**Appreciation of math aesthetic of student constructions**	Oooooooh!	Agg	
27:23	**Interesting quality of incorrect s. contribs.**	That's an interesting one.	Ag	
27:40	**More than one correct s. contrib**	Find some other ones that went through	Ag	
28:42	**Student finds non-standard example**	S: Hey wait. Can we put 4 times, paranthesis	Ag	
28:49	**Teacher moving toward a more general family of correct responses**	T: so you're saying that if i do the exact same thing with all of these... S: yes	Agg	

29:17	**Interesting quality of incorrect s. contribs.**	But it's an interesting equation... Why do you think they thought this would work?	Ag	Emphasizes that beauty is not just determined by correctness, but that correctness is also determined by beauty. (Papert, 19?) Carpenter's (1988) work on "teaching as problem solving" resonates with our intuition about the real mathematical challenges that mathematics teachers face in the context of teaching, but it also implies that for the constructive space of mathematics creation, there exists an analogous creative space for the creation of mathematics pedagogy, which might also be informed by the aesthetics of mathematics.
32:30	**Generalization**	What's the simplest form of the equation?		
33:50	**T gives rule for x-y coordinate**	I want you to move your cursor, keep your original x but now i want your y to be two less than your x.	Ag	
35:28	**Teacher questions about appearance of upfront space**	T: what do you see happening on the screen up there.	Agg	
38:16	**T gives a challenge to students**	T: let's make an equation that fits these lines now.	Ag	
46:54	**T question about math concept**	T: What does it mean to multiply two lines together? S: 'foil' thing		Allows t. to think about math ideas in new ways (visually); how visual representation matches the alg. notation.

49:48	**T gives a challenge to students**	T:...multiply your two y's together...		
55:56	**Appeal of visual representation of math concept**	T: it's looking pretty good up there	Agg	
56:17	**T questions about the appearance of upfront space**	T: she's saying it's looking like a parabola...	Agg	
1:00:43	**T gives challenge to st.**	T: some of you have an idea about an equation...		
1:02:53	**Appreciation of math aesthetic of student constructions**	T: that was a pretty good one	Ag	
1:03:50	**T question about math concept**	T: What did we graphically do here?	Agg	The math is embodied in the pictures and t and s understanding of what it represents.
1:04:51	**Emotional response to math concept**	T: it seems strange but...		Envision a use for hubnet that highlights the aesthetic qualities of mathematics by creating pedagogical artifacts of math. instruction that can be critiqued through a framework of mathematical beauty.

Tape u04011501 (Interview)

Time	Concept	**Quote**	**Level**	**Description**
31:00	Choosing "cool" topic	"…based on Pythagorean triples"	Agg	
32:10	Choosing " next" topic	"next thing …they come up with the standard form of a circle…"	Ag	
33:35	Welcoming attitude of alternate explanation of math. concept	"I am glad I got to see it this way, because…"		
34:10	Assessment of students' understanding	"So the kids got that pretty darn quick…"		
33:40	Seeing math concepts as dynamic or as a set of possibilities	"Exactly. So this length, it doesn't matter where it is, will always be…"	Agg?	
34:18	Pedagogy of possible cases	"Start with a bunch of points and say is this on the border of the orchard, inside or outside…"	Agg	From the "Orchard" section in IMP.
34:39-35:50	Class activity as inductive process converging on math generality	"So pretty soon they realized that..."	Ag?	
1:02:41	Choosing possible topics for the hubnet. How?	"I'm wondering if we can; I still want to do some stuff with maybe graphing..."		

Tape u04021601 (interview)

Time	Concept	**Quote**	**Level**	**Description**
0:00	**Teacher trying to understand derivatives (proof)**			
4:32	**Pedagogical criticism**	A: notice this tells you nothing. M: It tells you absolutely nothing about it.	Agg	Critique of an activity in IMP3 on informal study of derivatives
12:02	**Teacher trying to understand derivatives (proof)**	I don't have any idea why	Agg	
15:30	**Pedagogical criticism of self**	That's what i should have done		Value judgment about the effectiveness of instruction
15:30-	**Researcher presents algebraic explanation**			
16:56	**Researcher pedagogical criticism of self**	Now I don't know if that explains why.	Agg	
17:04	**(Critique) comparison of pedagogy**	So basically this is truly the understanding of it, and...		
18:22	**Pedagogical critique**	That helps me see where this is connected to mine.		
18:36	**Critique of future directions (activities) for pedagogy**	It might help for me to do this with other functions	Ag	
25:10	**Pedagogical critique**	I like that they get to experience this whole concept of derivatives		

32:52	**Appeal of proof for understanding math**	And I need the proof before		Artistry as explanation.
47:10	**Appeal for understanding "why"**	Why does it approach	Ag	
47:40	**Preference for inductive reasoning to understand**	I need to do this with three consecutively small numbers		

Tape u04011604 (classroom observation)

Time	Concept	**Quote**	**Level**	**Description**
0:00	**See u04011601**			Function activity
19:04	**T gives s a challenge**	Move somewhere on this graph where your x is a positive and your y is a positive number.	Ag	Aesthetic of mathematical idea like quadrant is that order is formed from seeming chaos with infinitely many solutions all from a particular family. This interplay between local individuality and global sameness is the essence of mathematical aesthetic. The teacher's ability to see how s. unique ways of thinking is unified in a coherent mathematics idea. This aesthetic builds up from individuality to a generalization that preserves the aesthetic of peculiarity.
24:34	**T focuses attention on upfront space**	So we've all made it to where y is positive and x is negative. Where is it on my graph here?	Agg	
27:32	**T gives s a challenge**	This time I want you to go some place where your x and your y are the exact same number.	Ag	The methodological pursuit is to find habits of teacher practice that are structurally analogous mathematical artistry in terms of a general notion of aesthetics in mathematics practices.
27:54	**T limits responses to challenge with counterexample.**	I don't mean negative and positive.	Ag	

28:50	**T questions s about the upfront space**	T: What's happening up there? What kind of a pattern are we making when we go to where our x and y are the same? S1: (gesturing with hand in the air) S2: we all on line.	Agg	
29:05	**T questions s response about the upfront picture**	Are we all on a line? (Pointing at upfront screen) Do you think everybody made it to where their x and y are the same?	Agg	In the appendix have a clips of T to illustrate the dynamic nature of some of the mathematical ideas.
29:56	**S notices a unified structure of s activity**	S3: we're all on a diagonal	Agg	
30:29	**T gives s a challenge**	I want you to move your car to your point.	Ag	Triangle to match stick function
				For example of individual motion creating a unified picture see how f(x) + c adds a constant to each y value of a function thereby causing a uniform shift in the entire graph of the function. Other examples include... These examples serve to illustrate the both the structural and aesthetic qualities of object/aggregate
33:12	**T directs s attention to upfront screen**	Well, it looks like some people are starting to make some kind of pattern.	Agg	

33:12	**T use of visual pattern to describe math concept**		Agg	
34:24	**T use of visual pattern to explore math concept**	What's happening with this shape? Do you think we're getting a shape?	Agg	
36:04	**T test points on or off the pattern**	Let's test them.	Ag	
37:25	**T places alg. rule secondary to visual pattern.**	Oh, we didn't talk about our rule; did we? Who came up with a rule for our table?	Agg	T uses algebra as a secondary explanation (proof) of the visual pattern.
38:29	**T acknowledges different correct solutions**	T: You did. You had that rule except you said x plus one plus x, right?	Ag	
38:42	**T test points on or off the pattern**	We better get –7 for her y.	Ag	How powerful are visual images in arousing aesthetic sensibilities for mathematics? What role does technology play in arousing mathematics aesthetic? (Is it the visuals?) What advantages exist (in terms of aesthetic perceptions)in displaying mathematics ideas visually?

Tape u03111801 (interview)

Time	Concept	**Quote**	**Level**	**Description**
4:25	**Ideas for part sims**	Using geometric patterns		
5:20	**Focus on pattern recognition in activities (induction)**	It's a pattern that now...	Agg?	Using patterns with many examples as a way to begin to think about part-sims and generative acts
11:11	**Aesthetic appeal of pattern (rules)**	Let's divide by two...	Agg	
11:28	**"interesting" (marker for mathematical aesthetic)**	Oh that's interesting...	Ag?	
13:40	**Pedagogical defense (modeling math practice)**	This first patterns unit is kind of to get them...to start looking at math differently		
22:07	**Allusion to math utility**	Maybe it's useful because...		
22:30	**Aesthetic purpose over utility**	A: I noticed that you chose to look for the patterns in the i's...	Ag?	Teacher makes decisions based upon aesthetic and utility of math
23:00	**Ideas for part sims**	A: so how can we turn this into a part sims?		Proposed interview question: What advantages of hubnet relative to that which you like most about math?
26:23	**Aesthetic appeal of students' pattern recognition**	We don't need the table...I'm doing this every time...	Agg?	
31:50	**Ideas for partsims:**	class creates a function of number of diagonals per polygon	Agg	

35:00-45:00	**Joint problem solving between researcher and teachers (proof)**	N lines creates how many partitions of space?		
45:00-1:04:10	**Joint problem solving between researcher and teachers with the technology (proof)**			Using the table feature and graph feature to assess the nature of a pattern
49:50	**Proof**	A: There was some insight that you gained in the way that you were drawing it...	Ag	
1:03:43	**Elation of solving a math problem (proof)**	We got it.		
1:04:20	**Verification of solution (proof)**			
1:06:38	**Elation of math problem solving**	I love that (inaudible).		
1:07:51	**Predicting emotional responses of students**	They would enjoy seeing...		
1:08:55	**Idea for partsims**	Maybe we want to go with something like this.		
1:16:35	**Predicting student ability-level**	This would be harder to do than that will.		
1:16:45	**Elation of problem solving**	This was an awful lot of fun. (re: 35:00)		

Tape u04011602 (classroom observation)

Time	Concept	**Quote**		**Description**
0:00				Function activity (IMP2). Product of 2 lines = quadratic.
9:19	**T gives s challenge**	I want you to write an equation that goes through that dot.	Ag	
9:43	**T gives hint on challenge**	So let me give you a hint how you might start that.	Ag	
10:45	**T gives s challenge**	I want to try to come up with another one.	Ag	
11:07	**T gives hint**	What if I start with y equals something times x or divided by x [x divided by something] then what am I going to have to add...	Ag	
16:31	**T questions about s responses**	Ok. How many of you think you have an equation that goes through this line?	Ag	
18:08	**T appeals for "interesting" s responses**	Let's try to use our more interesting one, ok.	Ag	

27:38	**T questions s about predictions of upfront screen**	T: What do you think is going to happen when we all send our equations that go through this point up here? S: (making star gesture with hand.) It's going to make (inaudible).	Agg	
28:35	**T constrains the scope of s responses**	For what I want to do right now, I want them to be linear.	Ag	
32:19	**T questions about certain s responses in the upfront space**	How many of them (lines) with negative slopes.	Agg	
32:30	**Students respond emotionally to "interesting" responses**	S: ("high-five" gesture with another student)	Agg	Evidence of aesthetic in that it elicits an emotional response of pleasure. This is salient for me because of my Black epistemology. Here emotional reaction is paramount.
32:38	**T questions about previous s predictions**	So how many of you expected it to look like that?	Agg	
32:38	**T questions s about appearance of upfront screen**	So how many of you expected it to look like that?	Agg	

32:43-33:04	**T questions s about appearance of upfront screen**	S1: I thought it was going to be (inaudible) (gestures with hands in a v-shape) T: why do you think it isn't? S2: (inaudible) T: yeah and also because the easiest ones were positive.	Agg	Important to notice how the class discussion is centered around the appearance of the upfront screen; what does aggregate class activity look like?
33:06-33:52	**T questions s about appearance of upfront screen**	T: so does anyone else have any comments about this? S1: (inaudible) T: so you think it kind of has some representation or gives you the feeling of something collapsing? S2: yeah. S3: yes. S4: they are all going to one point. T: and these are all going to one point? S5: Well actually they T: in our picture it looks like they 're all kind of connected together but really in real life you are saying they ... S6: yeah but if you were to draw a picture (inaudible)	Agg	(note: labels for students are arbitrary and unique to each episode. Same labels across episodes does not necessarily denote same student.)

33:56	**T gives s challenge**	T: now I want to go back out and we are going to do this again for number four, right? Except I'm going to give you a different point to do this on.	Ag	
35:18	**T gives s challenge**	Do the same thing. I want you to write two equations for (–4,0).	Ag	
45:20	**T gives hint on challenge**	If you could just sketch your line on the graph.	Ag	
49:38 (48:00-50:00)	**T makes aesthetic judgment of individual s responses**	I'm seeing some interesting things.	Ag	
53:10	**T emotionally responds to upfront screen.**	Oh my goodness.	Agg	T sees upfront screen as depicting some aesthetically appealing math construct.
54:37	**T questions s about upfront screen**	So tell me what's the same about this one (referring to upfront screen).	Agg	?what makes this time better than the first?
54:55	**T makes aesthetic judgment of upfront screen**	Makes an interesting design.	Agg	
55:00	**T gives s challenge**	How about now we take your two interesting linear equations, I don't	Ag	
55:08	**T constrains s responses with aesthetic perception**	You don't get to use my boring ones. Ok?	Ag	?do you think students understand what you mean by the word "interesting

56:49	**T gives s challenge**	Write down, not these two, but your most interesting one (equation) for this (the point 3,0), your most interesting one for that (the point –4,0). We going to put this (product of the two linear equations) as our equations.	Ag	
57:05	**T questions s about predictions in upfront screen**	What do you think will happen? Any ideas?	Agg	
1:00:25	**T questions s about predictions in upfront screen**	T: So what is going to happen with this when everybody sends it up? S1: just a whole bunch of curves. T: a whole bunch of curves? S1: all going through the same two points. S2: yeah.	Agg	
1:01:57	**T emotionally responds to upfront screen.**	I my gosh! There we go!	Agg	
1:02:55	**T questions about appearance of upfront screen**	So what do you think these graphs that we all sent here, um, what do they have in common with the two linear equations that we multiplied together.	Agg	

1:04:31	**T explores open-ended issues from student conjectures**	Does anyone think there is anything else in common with these quadratics and the straight lines?	Ag?	Inductive reasoning from the various cases that students create.
1:06:23	**S shows emotional response**	S1: Wow! That's kind of funny (interrupted)	Ag?	?why do you think s was emotionally charged by this?
1:17:47	**Teacher gives challenge**	I want you to really write down what you think the parabola shares with the line[s]. What to they have in common?	Agg?	

Tape u04011301 (interview – hubnet planning session)

Time	Concept	**Quote**	**Level**	**Description**
2:30	**Plan hubnet activity**	Just say, "so everybody go to your coordinate and let's see what happens."	Ag	
2:50	**Give s challenge Have s make predictions about the look of upfront screen**	What does everybody think will happen if we all move our x and y (interrupted).	Agg	Describe the potency of math idea when it is possible to see it dynamically. How is this a part of epistemic seeing?
3:13	**Give s challenge**	R: go to where your x and y have a positive value? T: right. Do that then say, "where are we?"	Ag	
3:23	**Value (emotional) judgment about a math concept**	It's always bugged me that they [quadrants] went counter clockwise... Just a convention, so we all agree.		
6:45	**Give s challenge**	Then we're going to say, "go to where your x and y are the same number.	Ag	
6:53	**T predicts emotional response**	Oh my gosh! What did that make? How interesting!	Agg	
6:59	**Give s challenge**	Can anyone make there x and y the same go into another quadrant?	Ag	
7:06	**T predicts s response and difficulty**	People will try it here and go, "oh, no way."	Ag	

7:24	**T points s to pattern on upfront screen**	Ok, now let's go back to our pattern.	Agg	
7:30	**T gives s challenge**	...where x was the number of triangles and y was how many match sticks.	Ag	
7:35	**Have s make predictions about the look of upfront screen**	What do you think is going to happen.	Agg	
7:50	**T points s to pattern on upfront screen T predicts emotional response**	Let's see what happens. Ooh! Aw! It's a line.	Agg	
8:10	**T and R rehearse the hubnet activity**			
9:40	**Have s make prediction about graphs of other functions**	It's a line. Does that prove all in-out tables are [a] line[s]?	Agg?	Focus on visual (graphical) representation of math.
10:04		Describe the patterns that they make.	Agg	
20:00-46:00	**T and R practice hubnet activitiy**			
46:40	**Defense of pedagogy**	R: so you're going to have a star right around that point? T: Um huh. And we're going to get another star around the other point.	Agg?	

48:33-48:58	**Deciding the final look of the upfront screen**	T: it might be a little too much to see a star, star, and a quadratic star going every which way... It might be cleaner actually this way.	Agg?	
52:08	**Aesthetic as goal of math instruction**	T: That's beautiful. R: There it is. That's a star. T: That is simply beautiful. They'll figure it out. I'll just give them a few more hints.	Agg	Mathematical aesthetic (embodied in the "star" as pedagogical artifact) as the goal of instruction.
52:20	**Appeal of math aesthetic in artifacts**	R: Why is that beautiful? Why do we say beautiful when we see something like that? T: Well, I think it's creating a pattern, I guess. I can see all those lines (making starburst gesture with hands) filling in...	Agg	

58:40	**T keeps a partly, affective goal for instruction**	Especially math, if you're just right there on the edge and someone comes in and interrupts, it's like, 'alright, there's twenty minutes down the drain, because you know how it is. Trying to hold that train of thought to the breaking through point and then if it goes, it's like, 'oh god. And it's harder, It's even harder the second time through it seems Getting them back enthused into it and focused, so...		
1:00:10-	**R and T practice hubnet activity**			

1:04:56	**Appeal of mathematical aesthetic**	Oh, that's beautiful! Look at that, ooh! But it is pretty impressive that they all go right through the spot with the other guys as (inaudible). But they'll see that. They'll see that. R: will they see that? Because everybody's line is going to go away. T: but it will still be on their calculators.	Agg	Significance: Aesthetic as a design principle for mathematics pedagogy and technological design. A discussion about whether or students will see the lines on their own calculators. Designing instruction for a generative classroom network seems force teacher to consider learning in terms of a final artifact. It is evident that T wants the students to be able to see a "design" in their respective screens, not for the design's sake but the mathematical concept that the design represents.
1:09:31	**Planning with aesthetic of upfront screen in mind**	R: Isn't that kind of like the one [equation] you just did? T: well, you know mine would have been steeper. It's not exactly. Ooh, look at those bad boys.	Ag?	Mathematics idea arrived at by the appearance of the upfront screen.
1:16:45		For the equations that go through (-4,0) i need to have key students make it one half x, y equal blank x times one eigth, right?	Agg?	
1:17:50		I would say before anyone sends their equations, i would like them to predict what's going to happen.	Agg	

1:19:00		R: what similarities [between the graph of the parabola and the lines] id you want to point out?	Agg	Mathematics idea arrived at by the appearance of the upfront screen.
1:20:03	**Generative activity as data to make generalizations**	I don't know if they're going to have enough data to collect that information.	Ag?	
1:20:25		R; What connections do you want them to see? That is my question. T: well i like all of those. Well the linear equations that make the quadratic go through the roots of the quadratic. So they cross the x-axis at the same points. The quadratic is the product of the two lines.	Ag?	
1:24:22		The coefficients (slope) in the two linear equations would affect whether it was inverted or not.	Ag?	

1:26:00		They'll either notice it or not and if they don't I might just have to say, "Well, does it make a difference? Or like did anyone have one of their linear equations have a negative slope?"	Ag?	
1:26:45		R: i wonder if there is any value in not multiplying the two (linear equations) together. T: Yeah. I decided that already, mainly because, i thought, well, practice is fine, but i don't want it to eat up our time.	Ag	
1:28:20	**T articulates a notion of aesthetic**	T: we will get some definite "cools"...	Agg	Describing mathematical aesthetic with a choreographed dance .metaphor.

Tape u04011601 (classroom observation)

Time	Concept	**Quote**		**Description**
0:00				Function activity
16:45	**T gives challenge to s.**	I want you to move to a place on the screen where	Ag	
17:45	**T focuses s. attention on the screen**	Look where everybody's going.	Agg?	
19:38	**T. questions class about screen**	Where does it look like everybody's going	Agg	What are math ped advantages to having upfront screen?
21:00	**T gives challenge to s.**	Go where your x and y are negative	Ag	
26:40	**T. questions class about screen**	Do you think everybody has gotten there (where x and y are negative)?	Agg	Question: how can you probe students to help them to see what you see aesthetically about math?
27:34	**T. questions class about screen**	What about that whit car? Do you think that is in a place where x is positive and y is negative?	Agg	
28:17	**T. questions class about screen**	T: What do you think x and y are going to be here? S: Because it's the opposite of the other one.	Agg	
33:00	**T gives challenge to s.**	What i want you to do is	Ag	How many triangles are formed with each new line?
33:24	**T ask s to describe a pattern**	Describe the pattern of this table	Agg	Why describe in words? Why patterns?
39:42	**T responds to s question by making him draw more examples**	I don't know. You may have to draw it.	Ag	

39:54	**T encourages exploration with more than just trivial case**	Ok, it won't be very interesting if everybody picks one.	Ag?	T is drawn to that which is interesting. But why no mention of explicit definition to s of what constitutes "interesting"
40:56	**T encourages exploration**	Pick any x that you'd like between 4 and 20.	Ag	Creates a space for math exploration (generativity)
44:23	**T explains x-y coord system**	How many of you have ever seen it written like this?	Ag	Creates a space for math exploration (generativity)
47:31	**T questions s about upfront screen**	What do you think is going to happen if all of you go to the point where you chose...	Agg	Linear function as a unified picture of generative student activity
56:19	**T limiting s responses to build a pattern**	If you pick points and get points all over the screen we don't see a pattern. We don't see anything interesting mathematically	Ag?	
58:47	T has preconceived pattern of s activity in mind	I'm going to get these points and look at them and see if they're in the right place or not.	Agg	
1:00:31	**T questions s about upfront screen**	Does anyone see another pattern emerging from those?	Agg	
1:00:51	**T calls attention to the upfront screen**	So Milo is saying that he can start to see a line here.	Agg	
1:01:46	**T and s dialogue about the appearance of the upfront screen**	So you're saying that the little cars go up two over one, up two over one all the way?	Agg	If this really is a recreation of first discovery of a specific math idea then it is not trivial that it is the picture that actual picture of the math ideas is salient.

1:04:40	**T discusses incorrect answers**	If i draw six triangles there how many match sticks do i have?	Ag	The pedagogical importance of identifying s examples, which are off the pattern as connoisseurship. Helping s understand what is salient, aesthetically pleasing about the math and why by counterexample.
1:05:58	**T discusses "interesting" answers**	I have a question. What about this point over here in the negatives?	Ag	highlights the importance of context.
1:11:40	**S get excited by the unified picture (iconic mathematics) of their activity**	Wooooooooo!	Agg	S impressed that the line goes through most all of the points. S show appreciation for mathematical aesthetics. (defined as the appreciation for the fact that one algebraically contrived object could describe all of their diverse constructions) (beyond the scope of this study, student aesthetic.)
1:12:28	**T summarizes s activity in one unified math concept**	The goal here is that we realize, which most of you already just said to me, that when we have a rule that we make up...	Agg	

APPENDIX B

Structural Objects of EGT (by lesson)
Lesson$_1$: Family of Quadratics

Design Phase	Level of perspective	Structural Objects	Description of Function (in terms of projecting mathematical aesthetic)	Vignette or quote
Conception	Agent			
	1. Space-creating function	Expanding the space		31:10; 35:09; 38:39; 36:42; 41:04; 43:48
		Constraining the space		30:28
		Examining the "contour" of the space		30:00; 37:16
	2. Predicting student interaction with the space	Predicting		30:45; 31:43; 34:04; 35:40; 37:38; 50:20
	3. Diversity as a stated objective	Aesthetic appeal of unique examples		32:40; 40:53
		Encouraging diversity		41:04; 43:48
	4. Consideration of passion	Showing emotion for unique examples		32:40; 40:53
		Predicting student emotional response		47:43
	Aggregate	1. Refining the projected image		32:54
		2. Conceptualizing the image as social pedagogy		41:10
		3. Conceptualizing the image as mathematics construct	Understanding the projected image as embodying a mathematics construct. Understanding how a mathematics construct can be expressed as dynamic structure comprised of various parts.	45:50
Planning	1. Emergent Perspective	Agent-to-aggregate	Perceiving the aggregate image and mathematical idea as emerging from the activities of lower level agents.	1:17:50; 1:19:00

	2. Aggregate as a collection of agents (or activity)		Perceiving the aggregate as a collection of diverse cases.	1:19:00; 1:20:03
	3. Dynamic perspective (shifting downward)	Aggregate-to-agent	The appearance of the aggregate dictates specific agent actions as if making sure all of the agent varieties are represented in the aggregate. Aggregate is perceived as embodying and making salient the commonality in diverse cases.	1:09:31; 1:16:45
	4. Aggregate perspective	Aesthetic appeal (passion)	These were some of the most explicit aesthetically-charged quote.	5:08; 52:20; 1:04:56; 1:09:31
Implementation	1. Agent	Hinting	Teacher gives hints to all or individual students about the agent-level activity.	9:43; 11:07; 45:20
		Space-creating	Creating, delimiting (giving rules), expanding (additional attempts: 10:45; 16:31) and constraining the space for the agent-level activity.	9:19; 28:35; 35:18; 53:56; 55:00;
		Aesthetic appeal for unique examples	"interesting"; diversity of ideas; aesthetic is perceived in agent-level activity and artifacts of student mathematical thinking, not just in aggregate	18:08; 55:08; 49:381; 56:49; 1:06:23
		Passion	Emotions and passion are an integral part of the activity	49:38; 55:08; 1:06:23
	2. Aggregate	Predicting image	Teacher asks students to predict the aggregate image	27:38; 32:38; 32:43; 57:05; 1:00:25
		Aesthetic/ passion	Teacher and students show passionate appeal for aggregate image	32:30; 49:38;
	3. Emergent perspective	Perspective is fluid and dynamic, changing from agent to aggregate; perceiving the aggregate as emerging from many various cases		32:19; 54:37
	4. Dynamic perspective	Perspective is fluid and dynamic, changing from aggregate to agent; perceiving aggregate as embodying the commonality of diverse cases; also focusing on aggregate as being made up of moving agents.		33:06; 1:02:55; 1:04:31; 1:17:47

Lesson$_2$: Rule for sets of points

Instructional design Phase	Level of perspective	Structural Objects	Description of Function (in terms of projecting mathematical aesthetic)	Vignette or quote	
				u04011501	u03112001
Conception (u04011501 and u03112001)	1. Agent: Space-creating function	1. Defining, constraining, and delimiting the space for agent-level activity			15:15
		2. Aesthetic appeal of diverse agents		33:35	
	2. Aggregate	Aesthetic appeal of the image of the aggregate		31:00	4:56; 19:52
	3. Dynamic perspective	Teacher perceives the aggregate as a composite structure of moving (movable) parts. Aggregate is perceived as a collection of agents. Aggregate is seen to embody the one commonality of a collection of diverse objects.		33:40 34:18	5:36
	4. Emergent perspective	Aggregate is perceived as an emerging pattern of agent behavior		34:39	5:36; 7:24; 18:25
				u04011301	u03112001
Planning (u04011301)	1. Agent	1. Space-creating functions		2:30; 3:13; 6:45; 6:59	8:00
		2. Predicting	Predicting potential caveats in the agent activity. Potential student difficulty with the agent-level activity. Navigating the contours of the space for agent activity	7:06	7:30
	2. Aggregate	Predicting the aggregate image		2:20; 7:35	
		Aesthetic/ passionate for the aggregate		6:35; 7:50	
	3. Beyond aggregate	Teacher asks students to predict beyond the aggregate artifact of the present lesson		9:40	
	4. Emergent perspective			7:24; 10:40	
				u04011603	u04011601
Implementation (**u04011603** and **u04011601**)	1. Agent	Delimit space	Gives the rules that defines the boundaries of the space for student agent-level activity; gives counter-examples; wrong answers	26:20; 28:57; 38:36	16:45; 21:00; 33:00; 40:56
		Expand space	Broadens the space of activity by increasing the number of possibilities for viable student responses; explains the possibilities and features of the space; give		39:42; 39:54; 40:56; 44:23; 1:04:40

			examples and counter-examples		
		Give hints			39:42
		Aesthetic/ passion for diverse agents		32:58; 39:29	
	2. Aggregate	Predicting the aggregate image	Teacher has students predict the final aggregate image		47:31
		Aesthetic/passion for aggregate			1:11:40
	3. Dynamic perspective	Perceiving the aggregate image as embodying the commonality of a diverse collection of cases; using aggregate image to check validity of agent behavior		38:30; 49:24	26:40; 27:34; 28:17; 58:47; 1:01:46; 1:12:28
	4. Emergent perspective			35:24; 48:24	17:45; 19:38; 39:54; 56:19
	5. Beyond aggregate	Teacher asks students to make inferences beyond scope of the present activity		48:50	

Lesson$_3$: Product of lines

Instructional design Phase	**Level of perspective**	**Structural Objects**	**Description of Function** (in terms of projecting mathematical aesthetic)	**Vignette or quote**
Implementation	1. Agent	Space-defining	Rules for the agent activity; defining by counter-example	11:32; 22:23; 33:50; 38:16; 49:48; 1:00:43
		Space-expanding		27:40; 28:42
		Contour of space		27:23; 28:40
		Aesthetic of unique cases; diversity		25:01; 28:42; 1:02:53
	2. Emergent perspective			13:41; 35:28; 56:17
	3. Aggregate	Aesthetic/ passion for aggregate image		55:56; 1:04:51
		Perceptions of aggregate image as embodying the mathematics		1:03:50

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